Star chromatic index

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Abstract

The star chromatic index $\chi'_s(G)$ of a graph G is the minimum number of colors needed to properly color the edges of the graph so that no path or cycle of length four is bi-colored. We obtain a near-linear upper bound in terms of the maximum degree $\Delta = \Delta(G)$. Our best lower bound on χ'_s in terms of Δ is $2\Delta(1+o(1))$ valid for complete graphs. We also consider the special case of cubic graphs, for which it is shown that the star chromatic index lies between 4 and 7. The proofs involve a variety of notions from other branches of mathematics and may therefore be of certain independent interest.

1 Motivation

Edge-colorings of graphs have long tradition. Although the chromatic index of a graph with maximum degree Δ is either equal to Δ or $\Delta + 1$ (Vizing [16]), it is hard to decide when one or the other value occurs. This is a consequence of the fact that distinguishing between graphs whose chromatic index is Δ or $\Delta + 1$ is NP-hard (Holyer [10]). This is true even for the special case when $\Delta = 3$ (cubic and subcubic graphs).

Two special parameters concerning vertex colorings of graphs under some additional constraints have received lots of attention. The first kind is that of an *acyclic coloring* (see [8, 2]), where we ask not only that every color class is an independent vertex set but also that any two color classes induce an acyclic subgraph. The second kind is obtained when we request that any two color classes induce a star forest — this variant is called *star coloring* (see [1, 15] for more details). These types of colorings give rise to the notions of the *acyclic chromatic number* and the *star chromatic number* of a graph, respectively.

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A *proper k-edge-coloring* of a graph G is a mapping $\varphi : E(G) \to C$, where C is a set (of *colors*) of cardinality k, and for any two adjacent edges e, f of G, we have $\varphi(e) \neq \varphi(f)$. A subgraph F of G is said to be *bi-colored* (under the edge-coloring φ) if $|\varphi(E(F))| \leq 2$. A proper k-edge-coloring φ is an *acyclic k-edge-coloring* if there are no bi-colored cycles in G, and is a *star k-edge-coloring* if there are neither bi-colored 4-cycles nor bi-colored paths of length 4 in G (by length of a path we mean its number of edges). The *star chromatic index* of G, denoted by $\chi'_{\mathrm{s}}(G)$, is the smallest integer k such that G admits a star k-edge-coloring. Note that the above definition of acyclic/star edge-coloring of a graph G is equivalent with acyclic/star vertex coloring of the line-graph $L(G)$.

If one considers the class of graphs $\mathcal{G}_\Delta$ of maximum degree at most Δ, Brooks' Theorem shows that the usual chromatic number is $O(\Delta)$. The maximum acyclic chromatic number on $\mathcal{G}_\Delta$ is $\Omega(\Delta^{4/3}/\log^{1/3}\Delta)$ and $O(\Delta^{4/3})$ (Alon, McDiarmid, and Reed [2]). The maximum star chromatic number on $\mathcal{G}_\Delta$ is $\Omega(\Delta^{3/2}/\log^{1/2}\Delta)$ and $O(\Delta^{3/2})$ (Fertin, Raspaud, and Reed [6]).

In contrast with the aforementioned $\Delta^{4/3}$ behaviour in the class of all graphs of maximum degree Δ, the acyclic chromatic index is linear in terms of the maximum degree. Alon et al. [2] proved that it is at most 64Δ, and Molloy and Reed [13] improved the upper bound to 16Δ. One would expect a similar phenomenon to hold for star edge-colorings. However, the only previous work [12] just improves the constant in the bound $O(\Delta^{3/2})$ from vertex coloring.

In this paper we show a near-linear upper bound for the star chromatic index in terms of the maximum degree (Theorem 3.1). Additionally, we provide some lower bounds (Theorem 4.1) and consider the special case of cubic graphs (Theorem 5.1). The proofs involve a variety of notions from other branches of mathematics and are therefore of certain independent interest.

2 Upper bound for $\chi'_{\mathrm{s}}(K_n)$

We shall first treat the special case of complete graphs. The study of their star chromatic index is motivated by the results presented in Section 3 since they give rise to general upper bounds on the star chromatic index.

Theorem 2.1 *The star chromatic index of the complete graph K_n satisfies*

$$\chi'_{\mathrm{s}}(K_n) \leq n \cdot \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\varepsilon > 0$ there exists a constant c such that $\chi'_{\mathrm{s}}(K_n) \leq cn^{1+\varepsilon}$ for every $n \geq 1$.

Proof. Let A be an n-element set of integers, to be chosen later. We will assume that the vertices of K_n are exactly the elements of A, $V(K_n) = A$, and color the edge ij by color $i + j$.

Obviously, this defines a proper edge coloring. Suppose that $ijklm$ is a bi-colored path (or bi-colored 4-cycle). By definition of the coloring we have $i + j = k + l$ and $j + k = l + m$, implying $i + m = 2k$. Thus, if we ensured that the set A does not contain any solution to $i + m = 2k$ with $i, m \neq k$ (in other words, A does not contain 3-term arithmetic progressions), we would have found a star edge-coloring of K_n.

We will use a construction due to Elkin [5] (see also [7] for a shorter exposition) who has improved an earlier result of Behrend [4]. As shown by Elkin [5], there is a set $A \subset \{1, 2, \ldots, N\}$ of cardinality at least $c_1 N (\log N)^{1/4} / 2^{2\sqrt{2}\sqrt{\log N}}$ such that A contains no 3-term arithmetic progression.

The defined coloring uses only colors $1, 2, \ldots, 2N$ (possibly not all of them), thus we have shown that $\chi'_s(K_n) \leq 2N$. We still need to get a bound on N in terms of n. In the following, $c_1, c_2, \ldots$ are absolute constants.

For every $\varepsilon > 0$ we have

$$n = |A| \geq c_1 N \frac{(\log N)^{1/4}}{2^{2\sqrt{2}\sqrt{\log N}}} \geq c_2 N^{1-\varepsilon} \tag{1}$$

Since we also have $N \leq c_3 n^{1+\varepsilon}$ for every $\varepsilon > 0$, we may plug this in (1) and use the fact that $(\log N)^{1/4}\, 2^{-2\sqrt{2}\sqrt{\log N}}$ is a decreasing function of N for large N to conclude that

$$n \geq c_4 N ((1+\varepsilon)\log n)^{1/4}\; 2^{-2\sqrt{2}\sqrt{(1+\varepsilon)\log n}}.$$

Thus we get $N \leq c_5 n 2^{2\sqrt{2}\sqrt{(1+\varepsilon)\log n}} (\log n)^{-1/4}$. One more round of this 'bootstrapping' yields the desired inequality

$$N \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

□

Remark. A tempting possibility for modification is to use a set A (in an arbitrary group) that contains no 3-term arithmetic progression and $|A + A|$ is small. Any such set could serve for our construction, with the same proof. Even more generally, we only need a symmetric function $p : A \times A \to N$, where $A = \{1, 2, \ldots, n\}$, such that $p(a, \cdot) : A \to N$ is a 1-1 function for each fixed $a \in A$, N is small, and p does not yield bi-colored paths (for all i, j, k, l, m we either have $p(i, j) \neq p(k, l)$ or $p(j, k) \neq p(l, m)$). We have been unable, however, to find a set that would yield a better bound than that of Theorem 2.1.

3 An upper bound for general graphs

The purpose of this section is to present a way to find star edge-coloring of an arbitrary graph G, using a star edge-coloring of the complete graph K_n with $n = \Delta(G) + 1$.

We will use the concept of *frugal colorings* as defined by Hind, Molloy and Reed [9]. A proper vertex coloring of a graph is called *β-frugal* if no more than β vertices of the same color appear in the neighbourhood of a vertex. Molloy and Reed [13, 14] proved that every graph has an $O(\log\Delta/\log\log\Delta)$-frugal coloring using $\Delta + 1$ colors. If Δ is large enough, one may use 50 for the implicit constant in the $O(\log\Delta/\log\log\Delta)$ asymptotics.

Theorem 3.1 *For every graph G of maximum degree Δ we have*

$$\chi'_s(G) \le \chi'_s(K_{\Delta+1}) \cdot O\Big(\frac{\log\Delta}{\log\log\Delta}\Big)^2 \tag{2}$$

and therefore $\chi'_s(G) \le \Delta \cdot 2^{O(1)\sqrt{\log\Delta}}$.

Proof. Using the above-mentioned result of Molloy and Reed [14], we find a β-frugal $(\Delta+1)$-coloring f with $\beta = O(\log\Delta/\log\log\Delta)$. We assume the colors used by f are the vertices of $K_{\Delta+1}$, so that the frugal coloring is $f : V(G) \to V(K_{\Delta+1})$. Let c be a star edge-coloring of $K_{\Delta+1}$. A natural attempt is to color the edge uv of G by $c(f(u)f(v))$. This coloring, however, may not even be proper: if a vertex v has neighbours u and w of the same color, then the edges vu and vw will be of the same color. To resolve this, we shall produce another edge-coloring, with the aim to distinguish these edges; then we will combine the two colorings.

We define an auxiliary coloring g of $E(G)$ using $2\beta^2$ colors. Let us first set

$$V_i = \{v \in V(G) : f(v) = i\}, \qquad i \in V(K_{\Delta+1})$$

and define the induced subgraphs $G_{ij} = G[V_i \cup V_j]$. For each pair $\{i,j\}$ we shall define the coloring g on the edges of G_{ij}; in the end this will define $g(e)$ for every edge e of G. Recall that the frugality of f implies that the maximum degree in G_{ij} is at most β. Consequently, the maximum degree in the (distance) square of $L(G_{ij})$ is at most $2\beta(\beta-1) < 2\beta^2$. Therefore, we can find a coloring of $E(G_{ij})$ using $2\beta^2$ colors so that no two edges of this graph have the same color, if their distance in the line graph is 1 or 2.

Now we can define the desired star edge-coloring of G: we color an edge uv by the pair

$$h(uv) = (c(f(u)f(v)), g(uv))\,.$$

First, we show this coloring is proper. Consider adjacent (distinct) edges vu and vw. If $f(u) \ne f(w)$, then $f(u)f(v)$ and $f(v)f(w)$ are two distinct adjacent edges of $K_{\Delta+1}$, hence c assigns them distinct colors. On the other hand, if $f(u) = f(w) = i$ (say), we put $j = f(v)$ and notice that uv and vw are two adjacent edges of G_{ij}, hence the coloring g distinguishes them.

It remains to show that G has no 4-path or cycle colored with two alternating colors. Let us call such object simply a *bad path* (considering C_4 as a closed path). Suppose for a contradiction that the path $uvwxy$ is bad. By looking at the first coordinate of h we observe that the c-color of the edges of the trail $f(u)f(v)f(w)f(x)f(y)$ assumes either just one value or two alternating ones. As c is a star edge-coloring of $K_{\Delta+1}$, this trail cannot be a path (nor a 4-cycle). A simple case analysis shows that in fact $f(u) = f(w) = f(y)$ and $f(v) = f(x)$. Put $i = f(u)$, $j = f(v)$ and consider again the g coloring of G_{ij}. By construction, $g(uv) \neq g(wx)$, showing that $uvwxy$ is not a bad path, a contradiction. □

As we saw in this section, an upper bound on the star chromatic index of K_n yields a slightly weaker result for general bounded degree graphs. We wish to note that, if convenient, one may start with other special graphs in place of K_n, in particular with $K_{n,n}$. It is easy to see that

$$\chi'_{\mathrm{s}}(K_{n,n}) \leq \chi'_{\mathrm{s}}(K_n) + n$$

(if the vertices of $K_{n,n}$ are a_i, b_i $(i = 1, \dots, n)$ then we color edges a_ib_j and a_jb_i using the color of the edge ij in K_n, while each edge a_ib_i gets a unique color). On the other hand, a simple recursion yields an estimate

$$\chi'_{\mathrm{s}}(K_n) \leq \sum_{i=1}^{\lceil \log_2 n \rceil} 2^{i-1} \chi'_{\mathrm{s}}(K_{\lceil n/2^i \rceil, \lceil n/2^i \rceil}) \,.$$

From this it follows that if $\chi'_{\mathrm{s}}(K_{n,n})$ is $O(n)$ (or $n(\log n)^{O(1)}$, $n^{1+o(1)}$, respectively) then $\chi'_{\mathrm{s}}(K_n)$ is $O(n \log n)$ (or $n(\log n)^{O(1)}$, $n^{1+o(1)}$, respectively).

4 A lower bound for $\chi'_{\mathrm{s}}(K_n)$

Our best lower bound on $\chi'_{\mathrm{s}}(K_n)$ is provided below and is linear in terms of n. The upper bound from Theorem 2.1 is more than a polylogarithmic factor away from this. So, even the asymptotic behaviour of $\chi'_{\mathrm{s}}(K_n)$ remains a mystery.

Theorem 4.1 *The star chromatic index of the complete graph K_n satisfies*

$$\chi'_s(K_n) \geq 2n(1 + o(1)).$$

Proof. Assume there is a star edge-coloring of K_n using b colors. Let a_i be the number of edges of color i, let $b_{i,j}$ be the number of 3-edge paths colored

i, j, i. We set up a double-counting argument. Note that all sums over i, j are assumed to be over all available colors (that is, from 1 to b). As every edge gets one color, we have

$$\sum_i a_i = \binom{n}{2}. \tag{3}$$

Fixing i, we have a matching M_i with a_i edges and each edge sharing both ends with an edge from M_i contributes to some $b_{i,j}$. Consequently,

$$\sum_j b_{i,j} = 4\binom{a_i}{2}. \tag{4}$$

Finally, we fix color j and observe that each 3-edge path colored i, j, i (for some i) uses two edges among the $2a_j \cdot (n - 2a_j)$ edges connecting a vertex of M_j to a vertex outside of M_j. This leads to

$$\sum_i b_{i,j} \leq a_j(n - 2a_j). \tag{5}$$

Now we use (4) and (5) to evaluate the double sum $\sum_{i,j} b_{i,j}$ in two ways, getting

$$4\sum_i \binom{a_i}{2} \leq \sum_j a_j(n - 2a_j).$$

This inequality reduces to

$$4\sum_i a_i^2 \leq (n+2)\sum_i a_i.$$

By the Cauchy-Schwartz inequality, $(\sum a_i)^2 \leq b \cdot \sum a_i^2$, and then using (3), we obtain

$$4\binom{n}{2} \leq b(n+2).$$

Therefore, $b \geq 2n(n-1)/(n+2) = (2 + o(1))n$. □

5 Subcubic graphs

A regular graph of degree three is said to be *cubic*. A graph of maximum degree at most three is *subcubic*. A graph G is said to cover a graph H if there is a graph homomorphism from G to H that is locally bijective. Explicitly, there is a mapping $f : V(G) \to V(H)$ such that whenever uv is an edge of G, the image $f(u)f(v)$ is an edge of H, and for each vertex $v \in V(G)$, f is a bijection between the neighbours of v and the neighbours of $f(v)$.

Theorem 5.1 (a) *If G is a subcubic graph, then $\chi'_{\mathrm{s}}(G) \le 7$.*

(b) *If G is a simple cubic graph, then $\chi'_{\mathrm{s}}(G) \ge 4$, and the equality holds if and only if G covers the graph of the 3-cube.*

For the part (a) of this theorem we will need the following lemma. It seems to be possible to use this lemma for other classes of graphs, therefore it might be of certain independent interest.

Lemma 5.2 *Let $f : E(G) \to \{1, \dots, k\}$ be a k-edge coloring.*

(a) *Let e be an edge of G. Suppose that the restriction of f to $E(G) \setminus \{e\}$ is a star edge-coloring of $G - e$ and that $f(e)$ is distinct from $f(e')$ whenever $d(e, e') \le 2$ (that is, either e, e' share a vertex, or a common adjacent edge). Then f is a star edge-coloring of G.*

(b) *Let A be a set of vertices of G, let $B = V(G) \setminus A$, and let X be the set of edges with one end in A and the other in B. Suppose that*

1. *(a restriction of) f is a star edge-coloring of $G[A]$;*
2. *(a restriction of) f is a star edge-coloring of $G[B]$;*
3. *no edges e_1, e_2 in X share a common vertex in A or a common adjacent edge in $G[A]$;*
4. *for every edge $e \in X$ and every edge e' in $G[B] \cup X$ such that $d(e, e') \le 2$ we have $f(e) \ne f(e')$ (distance is measured in $G[B] \cup X$, not in G);*
5. *for every edge $e \in X$ and every edge e' in $G[A]$ we have $f(e) \ne f(e')$.*

Then f is a star edge-coloring of G.

Proof (of the lemma). (a) Since f is a star edge-coloring of $G - e$, no 4-path (or 4-cycle) in $G - e$ is bi-colored. If P is a bi-colored 4-path (4-cycle) containing e, then P contains an edge of the same color as e at distance ≤ 2 from e, a contradiction.

(b) Conditions (3), (4), (5) imply that for every edge $e \in X$ and every edge $e' \in E(G)$, if $d(e, e') \le 2$, then $f(e) \ne f(e')$. Therefore, we can repeatedly apply part (a), starting with the graph $G[A] \cup G[B]$ and adding one edge of X at a time. $\square$

To explain a bit the conditions of part (b) in the above lemma: the point here is that in the condition 5, we do not check what is the distance of e and e'. In our applications, A will be a particular small subgraph of G (such as those in Figure 1) and B the 'unknown' rest of the graph. We do not want to distinguish whether some edges in X share a vertex in B. This, however, may create new 4-paths, henceforth the particular formulation of this lemma.

Proof (of the theorem). (a) Trying to get a contradiction, let us assume that G is a subcubic graph with the minimum number of edges for which $\chi'_{\mathrm{s}}(G) > 7$.

We first prove several properties of G (connectivity, absense of various small subgraphs). This finally allows us to construct the desired 7-edge-coloring by decomposing G into a collection of cycles connected by paths of length 1 or 2.

Clearly, G **is connected. Suppose that** G **contains a cutedge** xy**.** Let G_x and G_y be the components of $G - xy$ which contain the vertex x and y, respectively. By the minimality of G, each of G_x and G_y admits a star 7-edge-coloring. In G_x there are at most 6 edges that are incident to a neighbor of x. By permuting the colors, we may assume that color 7 is not used on these edges. Similarly, we may assume that color 7 is not used on the edges in G_y that are incident with neighbors of y. Then we can color the edge xy by using color 7 and obtain a star 7-edge-coloring of G. This contradiction shows that G **is 2-connected.**

If G contained **a path** $wxyz$**, where** x **and** y **are degree-2 vertices,** then we could color $G - xy$ by induction, and extend the coloring to a star 7-edge-coloring of G by using Lemma 5.2. (For the edge $e = xy$ we use a color that does not appear on the at most six edges incident to w or z.) Thus, such a path $wxyz$ does not exist. In particular, G is not a cycle.

Suppose next that G contains **a degree-2 vertex** z **whose neighbors** x **and** y **are adjacent.** We will use Lemma 5.2 for $e = xz$. By induction we may find a star edge-coloring of $G - e$, and as there are at most six edges in G at distance ≤ 2 from e, we can extend the coloring to e to satisfy the condition in Lemma 5.2, part (a). So the graph G can be star edge-colored using 7 colors, a contradiction. This shows that the neighbors of a degree-2 vertex cannot be adjacent in G.

Further suppose that G contains **parallel edges.** Three parallel edges would constitute the whole (easy to color) graph, so suppose there are two parallel edges between vertices u and v. Unless G contains a bridge, or G has at most three vertices (and is easy to color), there are neighbors u' of u, v' of v and $u' \neq v'$. By induction we can color $G \setminus \{u, v\}$. Next, we extend this coloring to the edges uu', vv', so that each of them has different color that the ≤ 6 edges at distance ≤ 2 from it. Now we distinguish two cases. If uu' and vv' have diffent colors, say a and b, then it is enough to use on the two parallel edges any two distinct colors that are different from a and b. If uu' and vv' have the same color, then there are at most 5 colors of edges at distance ≤ 2 from the parallel edges, so we may use Lemma 5.2, part (a).

Next we suppose that G contains **one of the first three graphs in Figure 1** as a subgraph, where other edges of G attach only at the vertices denoted by the empty circles, and some of these vertices may be identified. We use Lemma 5.2, part (b). We let A be the set of vertices of the subgraph in the figure that are denoted by full circles, so X is the set of the three thick edges. By induction, $G[B]$ is star 7-edge-colorable. This coloring can be extended to $G[B] \cup X$ so that color of each edge e in X differs from the color of all edges at distance ≤ 2 from e (there are at most 6 such edges). We assume that the colors used on X are in $\{5, 6, 7\}$. For $G[A]$ we use the coloring as shown in the figure. This satisfies conditions of Lemma 5.2, part (b), and therefore G can be star 7-edge-colored.

Next suppose that G contains **the fourth graph in Figure 1** as a subgraph (again other edges can only attach at the 'empty' vertices, some of which may be identified). We use Lemma 5.2, part (b) to show that $G - e$ is 7-edge colored in a particular way that allows us to use Lemma 5.2, part (a) to extend the coloring on e. We let A be the vertices of the pentagon, so that X is the set of the three thick edges. Note that the conditions of the part (b) are satisfied for $G - e$, but not for G itself. By induction there is a star 7-edge-coloring of $G[B]$, and we again extend it to X so that the edges in X have distinct color from edges at distance ≤ 2. Observe that there are at most six edges in $G[B] \cup X$ that are at distance ≤ 2 from e, so there is a color, say C, not used on any of those. We shall reserve C to be used at e. First, however, we apply part (b) to color $G - e$. We use the coloring of $G[A]$ shown in the figure, assuming that $C \notin \{1, 2, 3\}$ and that none of the colors $1, 2, 3$ is used on X. Finally, we use part (a) to extend the coloring on G, letting the color of e be C.

As the last reduction, we show that G **does not contain a path** $wxyz$**, where** w **and** z **are degree-2 vertices.** (We may assume that all vertices among w, x, y, z, and their neighbours are distinct, as otherwise G contains one of the previously handled subgraphs.) If G did contain such a path, we could color $H = G - \{w, x, y, z\}$. Next, we describe how to extend this coloring to a star 7-edge-coloring of G. We will denote the edges as in Figure 2; to ease notation we will use a to denote both the edge and its color.

The edges $a, \ldots, h$ are part of H, so they are colored already. Similarly as in the previous cases, we choose a color for s, t, u, v so that none of these edges shares a color with an edge of H at distance at most 2. (So that the color of s differs from the colors of a, b, and the at most 4 edges adjacent with a or b; note that it is not a problem if, e.g., s and t have a common neighbour $b = c$. However, it cannot happen that, say, s and t are adjacent, and so one of these neighbours at distance ≤ 2 has unassigned color.) Using 7 colors, this is easy to achieve. Next, we pick a color for q that differs from c, d, e, f, t, and u.

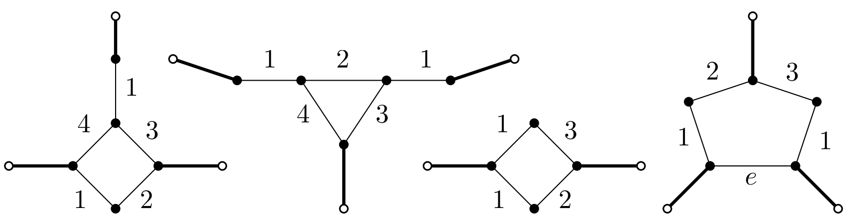

Figure 1: Subgraphs that cannot appear in a minimal counterexample.

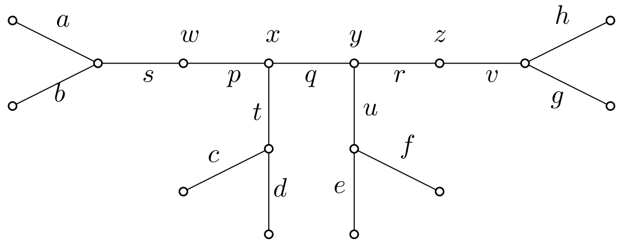

Figure 2: Illustration of the proof that minimal counterexample to Theorem 5.1 does not contain path $wxyz$ as depicted in the figure.

Now, we distinguish several cases based on colors of s, q, and v. We again assume the colors are 1, ..., 7; up to symmetry we have only the following cases.

Case 1. $s = 1, q = 2, v = 3$
We only need to avoid bi-colored paths $aspt$, $bspt$, $sptc$, $sptd$, and the four symmetrical paths in the right part of the figure. If $t = 1$, we choose p to be different from $1, 2, a, b, c, d$. If $t \neq 1$, it suffices to make p different from s, q, t. The procedure for r is analogous.

Case 2. $s = 1, q = 1, v = 2$
In this case $t \neq 1$, so we only need to avoid bi-colored paths $aspq, bspq, spqr, spqu$ and $urvh, urvg$, $eurv, furv$. If $u = 2$, we make sure that r differs from $1, 2, e, f, g, h$. Otherwise, it suffices to make r different from q, u, v. Then we choose p to differ from $a, b, 1, t, u, r$.

Case 3. $s = 1, q = 2, v = 1$
This is handled in exactly the same way as Case 1.

Case 4. $s = 1, q = 1, v = 1$
Now $t, u \neq 1$, so we only need to avoid the paths $aspq, bspq, spqr, pqrv, qrvg, qrvh$, and $spqu, tqrv$. To do this, we only need to ensure, that $p \neq 1, a, b, t, u, r$ and $r \neq 1, h, g, u, t, p$, which is easily possible. This finishes the proof of the claim that minimal counterexample G does not contain a path $wxyz$, where w and z are degree 2-vertices.

Let G' be the graph obtained from G by suppressing all degree-2 vertices, i.e., replacing each path xzy, where z is a degree-2 vertex, by the single edge xy. By the above, G' is a 2-connected cubic graph (without multiple edges). By a result of Kaiser and Škrekovski [11], G' contains a perfect matching M' such that M' does not contain all edges of any minimal 3-cut or 4-cut. Note that each edge in G' corresponds either to a single edge in G or to a path of length two. Let M denote the set of edges of G corresponding to an edge of M'. Our goal is to use four colors (say 4, 5, 6, 7) on M', and three colors (say 1, 2, 3) on the other edges that form a disjoint union of circuits. We form an auxiliary graph K, whose vertices are the edges in M. We make two of these edges e, f adjacent in K if they are either forming a 2-edge path corresponding to an edge in M' or if there is an edge in G joining an end of e with an end of f. Observe that K is a graph of maximum degree at most four. Also note that if K is disconnected, then each component contains a vertex of degree at most three. By the Brooks Theorem, K is 4-colorable unless it contains a connected component isomorphic to K_5. It is easy to see that the latter case occurs if and only if $K = K_5$ and $G = G'$.

Let us first consider the case when K is 4-colorable. In this case we will not need the fact that M' does not contain minimal 3-cuts or 4-cuts. The 4-coloring of the vertices of K determines a 4-coloring of the edges in M with the property that every color class is an induced matching in G. We shall show that we can star 3-color the edges in $G - M$ unless $G - M$ contains a 5-cycle; this case will be treated separately. By extending that 3-edge coloring to a 7-edge-coloring of G (by using the 4-coloring of edges in M) we obtain a star 7-edge-coloring since none of the four colors used on the edges in M can give rise to a bi-colored 4-path or a cycle (Lemma 5.2). Thus it suffices to find a star 3-edge-coloring of $G - M$. This is not hard unless $G - M$ contains a 5-cycle. Recall that $G - M$ is the union of disjoint cycles and every k-cycle, where $k \neq 5$, admits a star 3-edge-coloring: This is easy if $k \in \{3, 4\}$ or if k is divisible by three. If $k \equiv 1 \pmod 3$ and $k > 5$, we can use the colors in the following order $1232123 \cdots 123$. Similarly, if $k \equiv 2 \pmod 3$ and $k > 5$, we can use the colors $12132123 \cdots 123$. Thus, the only problem are the 5-cycles in $G - M$. To color them, we shall choose an edge $e = e_C$ in each 5-cycle C and a color $c = c_C$, that is otherwise used as a color for M. Then we color e with color c and color the 4-path $C - e$ as $1, 2, 3, 1$. We pick c and e in such a way that no edge of M at distance at most 2 from e has color c (we will show below that this is possible). It is easy to check that this, together with the fact that K is properly colored, prevents all 4-paths and 4-cycles from being bi-colored (Lemma 5.2 again). So, this finishes the proof of the case when K is 4-colorable—provided we show how to pick e and c for each 5-cycle C. To do this, we let F be the set of edges of M that are incident with a vertex of C but not part of C. Further, we let X be the (possibly empty) set of edges of M adjacent with some edge of F. Easily, $|X|$ is the number of 2-edge paths in M that are adjacent to C. (A 2-edge path with both ends at C counts twice; in this case X and C intersect.) As G contains no 3-edge path with both ends of degree 2, we have $|X| \leq 2$. We distinguish two cases based on the color pattern on edges of F. These cases cover all possibilities up to renaming the colors.

Case 1. Edges of F use in some order colors 4, 4, 5, 6, and 7 (that is, one color appears twice, the other colors once). If $X = \emptyset$, there are three possible choices for edge e: for each color c among 5, 6, and 7 we may choose the edge of C opposite to the edge of F colored c. Edges of X may be at distance 2 to some of these edges of C. However, there are at most two such edges, hence at most two colors are affected. So, one of colors 5, 6, and 7 is still valid.

Case 2. Edges of F use in some order colors 4, 4, 5, 5, and 6 (that is, two colors twice, one once, one not at all). In this case, if $X = \emptyset$, all five edges of C can be colored 7. Each edge of X (if such an edge exists) is at distance 2 from two edges of C, so one edge of C is far from edges of X and we can let this edge be e and c be 7.

Finally, let us consider the case when K does not admit a 4-coloring, i.e., $K = K_5$. As argued before, this implies that $G = G'$ is a cubic graph containing precisely 10 vertices. Note that $G - M$ is a 2-regular graph with no 3-cycles or 4-cycles (due to the choice of M'). Thus $G - M$ is isomorphic either to a 10-cycle, or to the union of two 5-cycles. Moreover, the edges of K are in bijective correspondence with $E(G - M)$. This correspondence is clear if we view K as the graph obtained from G by contracting the edges in M. We will prove that there is another perfect matching in G, whose auxiliary graph K admits a 4-coloring, unless G is the Petersen graph. Note that this other perfect matching may contain 3-cuts or 4-cuts, but as the auxiliary graph is 4-colorable, this does not harm us.

If $G - M$ is the union of two 5-cycles, then it is easy to check that G is the Petersen graph, and hence $\chi'_s(G) = 5$. (A star 5-edge-coloring is easy to find, and the star 4-edge-coloring does not exist as shown in part (b) below.)

The final case is that $G - M$ is a 10-cycle. Let us denote the vertices of G by integers i, i', $1 \le i \le 5$, so that $M = \{ii' \mid 1 \le i \le 5\}$. Let N_1 and N_2 be the two perfect matchings contained in $G - M$. Let us consider the closed walk in K corresponding to the 10-cycle $G - M$. This closed walk is an Eulerian trail T in $K = K_5$ (no edges are repeated). It is easy to check that each Eulerian trail in K_5 contains a closed subwalk of length 3. We may assume that this closed subwalk corresponds to the path $1231'$ in G. If its edges 12 and $31'$ are in N_1 (say), then we may choose the matching N_1 in place of M and define graph K' in the same way K was defined using M. The vertices of K' corresponding to 12 and $31'$ are "doubly adjacent" (they use two edges of $G - N_1$ to connect them). This doesn't leave enough edges to create K_5, so $K' \ne K_5$, and as argued before, K' is 4-colorable. This case was already settled without using the special properties of M. This completes the proof.

(b) Every 3-edge-coloring of a cubic graph has bi-colored cycles, thus $\chi'_s(G) \ge 4$. In Figure 3 there is a 4-edge-coloring of the cube Q_3. It is easy to verify that this is indeed a star edge-coloring. Perhaps the fastest way to see this is to observe that for each $i \ne j$, there is (a unique) 3-edge path colored i, j, i between the two vertices colored j. Consider now a graph G that covers Q_3 and use the covering map to lift the edge-coloring of Q_3 to an edge-coloring of G. From the definition of covering projections we see that a path of length 2 in G is mapped to a path of length 2 in Q_3. It follows that the defined edge-coloring is proper. It also follows that a path of length 4 in G is mapped to a path of length 4 in Q_3 or to a 4-cycle in Q_3, and a 4-cycle in G is mapped to a 4-cycle in Q_3. It follows that we have a star-edge coloring of G.

For the reverse implication, suppose that G has a star 4-edge-coloring c. Let us first define a (vertex) 4-coloring f by letting $f(v)$ be the (unique) color that is missing on edges incident with v.

f is a proper coloring. For a contradiction, suppose that $f(u) = f(v)$ for an edge uv of G. Let u_1, u_2 be the other neighbors of u, and v_1, v_2 be the other neighbors of v. By symmetry we may assume that $f(u) = f(v) = 4$, $c(uv) = 3$, $f(uu_i) = f(vv_i) = i$ (for $i = 1, 2$). The bi-chromatic paths u_iuvv_i imply that 3 is neither used on edges incident with v_1 nor on those incident with v_2. This, however, implies that there is an edge colored 2 incident with v_1 and an edge colored 1 incident with v_2, which create a bi-chromatic 4-edge path (or 4-cycle), a contradiction. Note that the cases where uv is contained in a triangle ($u_1 = v_2$, $u_2 = v_1$ or both) are also covered by the above.

f is a covering map $G \to K_4$. Suppose for a contradiction that there is a vertex v with neighbors v_1, v_2 such that $f(v_1) = f(v_2)$. By symmetry we may assume that $f(v) = 4$, $f(v_1) = f(v_2) = 3$, $c(vv_i) = i$. Now v_1 must be incident with an edge of color 2 and v_2 must be incident with an edge of color 1, producing again a bi-chromatic 4-edge path (or cycle).

f together with c define a covering $G \to Q_3$. Let i, j, k, l denote $1, 2, 3, 4$ in some order. If a vertex v of G has $f(v) = i$ then the c-colors of its incident edges are j, k, l and the same holds for the f-colors of its adjacent vertices. There are exactly two possibilities, either the edges incident with v colored j, k, l lead to vertices colored k, l, j (respectively), or to vertices colored l, j, k. We refer to these two possibilities as the *local color pattern* at v.

Observe that in Q_3 as depicted in Figure 3, there are for each i two vertices colored i and they use different local color patterns. This implies there is a unique vertex mapping $F : V(G) \to V(Q_3)$ such that for each $v \in V(G)$ the following conditions hold:

1. we have $f(v) = f(F(v))$ (we use f also for the vertex coloring of Q_3), and
2. v and $F(v)$ use the same local color pattern.

To show that F is a covering map, we need to observe that for each $v \in V(G)$, the three neighbours of v in G map by F to the three neighbours of $F(v)$ in Q_3. As we already know that f is a covering map to K_4, it suffices to show, that a neighbour u of v is indeed mapped to the neighbour of $F(v)$ with color $f(u)$ (and not to the other vertex with the same color). For this we observe that the local coloring pattern at a vertex v determines the local coloring pattern at each neighbour of v, in any cubic graph that is star 4-edge-colored. As this holds both in G and in Q_3, our definition of F yields a covering map, which finishes the proof. $\square$

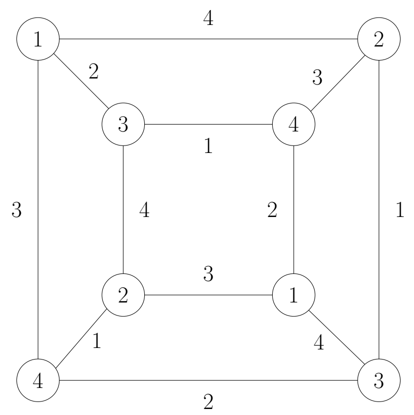

Figure 3: Cube Q_3 with star edge-coloring by four colors. The vertex labels are used in the proof of Theorem 5.1.

There are cubic graphs whose star chromatic index is equal to 6. One example is $K_{3,3}$. To see this, let us suppose that we have a star edge-coloring of $K_{3,3}$, and let F be a color class. If $|F| = 3$, then every other color class contains at most one edge and hence there are at least seven colors all together. So, we may assume that every color class contains one or two edges only. If $F = \{ab, cd\}$ is a color class, then one of the edges ad or cb forms a singleton color class since the second edge in the color class of ad (and the same for cb) would need to be the edge of $K_{3,3}$ disjoint from a, b, c, d. This implies that there are at least two singleton color classes. Hence, the total number of colors is at least 6. Finally, a star 6-edge-coloring of $K_{3,3}$ is easy to construct, proving that $\chi'_s(K_{3,3}) = 6$.

6 Open problems

As we saw in Sections 2 and 4, establishing the star chromatic index is nontrivial even for complete graphs. We established bounds

$$(2 + o(1)) \cdot n \leq \chi'_s(K_n) \leq n \cdot \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

Question 1 *What is the true order of magnitute of $\chi'_s(K_n)$? Is $\chi'_s(K_n) = O(n)$?*

In the previous section we obtained the bound $\chi'_s(G) \leq 7$ for a subcubic graph G. We also saw that $\chi'_s(K_{3,3}) = 6$. A bipartite cubic graph that we thought might require seven colors is the Heawood graph (the incidence graph of the points and lines of the Fano plane). However, it turned out that also its star chromatic index is at most 6. After some additional thoughts, we propose the following.

Conjecture 2 *If G is a subcubic graph, then $\chi'_{\mathrm{s}}(G) \leq 6$.*

It would be interesting to understand the list version of star edge coloring: by an edge k-list for a graph G we mean a collection $(L_e)_{e\in E(G)}$ such that each L_e is a set of size k. We shall say that G is *k-star edge choosable* if for every edge k-list (L_e) there is a star edge coloring c such that $c(e) \in L_e$ for every edge e. We let $ch'_s(G)$ be the minimum k such that G is k-star edge choosable. All of the results in this paper may have extension to list colorings. Let us ask specifically two questions:

Question 3 *Is it true that $\mathrm{ch}'_s(G) \leq 7$ for every subcubic G? (Perhaps even ≤ 6?)*

Question 4 *Is it true that $\mathrm{ch}'_s(G) = \chi'_s(G)$ for every G?*

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