3-choosability of planar graphs with (≤ 4)-cycles far apart

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Abstract

A graph is k-choosable if it can be colored whenever every vertex has a list of at least k available colors. We prove that if cycles of length at most four in a planar graph G are pairwise far apart, then G is 3-choosable. This is analogous to the problem of Havel regarding 3-colorability of planar graphs with triangles far apart.

1 Introduction

All graphs considered in this paper are simple and finite. The concepts of list coloring and choosability were introduced by Vizing [13] and independently by Erdős et al. [7]. A *list assignment* of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. An *L-coloring* is a function $\varphi : V(G) \to \bigcup_v L(v)$ such that $\varphi(v) \in L(v)$ for every $v \in V(G)$ and $\varphi(u) \neq \varphi(v)$ whenever u and v are adjacent vertices of G. If G admits an L-coloring, then it is *L-colorable*. A graph G is *k-choosable* if it is L-colorable for every list assignment L such that $|L(v)| \geq k$ for all $v \in V(G)$. The *distance* between two vertices is the length (number of edges) of the shortest path between them. The distance $d(H_1, H_2)$ between two subgraphs H_1 and H_2 is the minimum of the distances between vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

The well-known 4-color theorem (Appel and Haken [3, 4]) states that every planar graph is 4-colorable. Similarly, Grötzsch [8] proved that every

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triangle-free planar graph is 3-colorable. For some time, the question whether these results hold in the list coloring setting was open; finally, Voigt [14, 15] found a planar graph that is not 4-choosable, and a triangle-free planar graph that is not 3-choosable. On the other hand, Thomassen [10, 11] proved that every planar graph is 5-choosable and every planar graph of girth at least 5 is 3-choosable. Also, Kratochvíl and Tuza [9] observed that every planar triangle-free graph is 4-choosable.

Motivated by Grötzsch's result, Havel asked whether there exists a constant d such that if the distance between each pair of triangles in a planar graph is at least d, then the graph is 3-colorable. This question was open for many years, finally being answered in affirmative by Dvořák, Král' and Thomas [6] (although their bound on d is impractically large). Due to the result of Voigt [15], an analogous question for 3-choosability needs also to restrict 4-cycles: does there exist a constant d such that if the distance between each pair of (≤ 4)-cycles in a planar graph is at least d, then the graph is 3-choosable? We give a positive answer to this question:

Theorem 1. *If G is a planar graph such that the distance between each pair of (≤ 4)-cycles is at least 26, then G is 3-choosable.*

This bound is quite reasonable compared to one given for Havel's problem [6]. However, it is far from the best known lower bound of 4, given by Aksionov and Mel'nikov [2].

2 Proof of Theorem 1

For a subgraph H of a graph G, let $d(H) = \min_F d(H, F)$, where the minimum goes over all (≤ 4)-cycles F of G distinct from H. Let $t(G) = \min_H d(H)$, where the minimum goes over all (≤ 4)-cycles H of G. A *path of length k* (or a *k-path*) is a path with k edges and $k+1$ vertices. For a path or a cycle X, let $\ell(X)$ denote its length. Let r be the function defined by $r(0) = 0$, $r(1) = 2$, $r(2) = 4$, $r(3) = 9$, $r(4) = 13$ and $r(5) = 16$. For a path P, let $r(P) = r(\ell(P))$. Let $B = 26$. Using the proof technique of precoloring extension developed by Thomassen [11], we show the following generalization of Theorem 1:

Theorem 2. *Let G be a planar graph with the outer face C such that $t(G) \geq B$, and P a path such that $V(P) \subseteq V(C)$. Let L be a list assignment such that*

(S1) $|L(v)| = 3$ *for all* $v \in V(G) \setminus V(C)$;

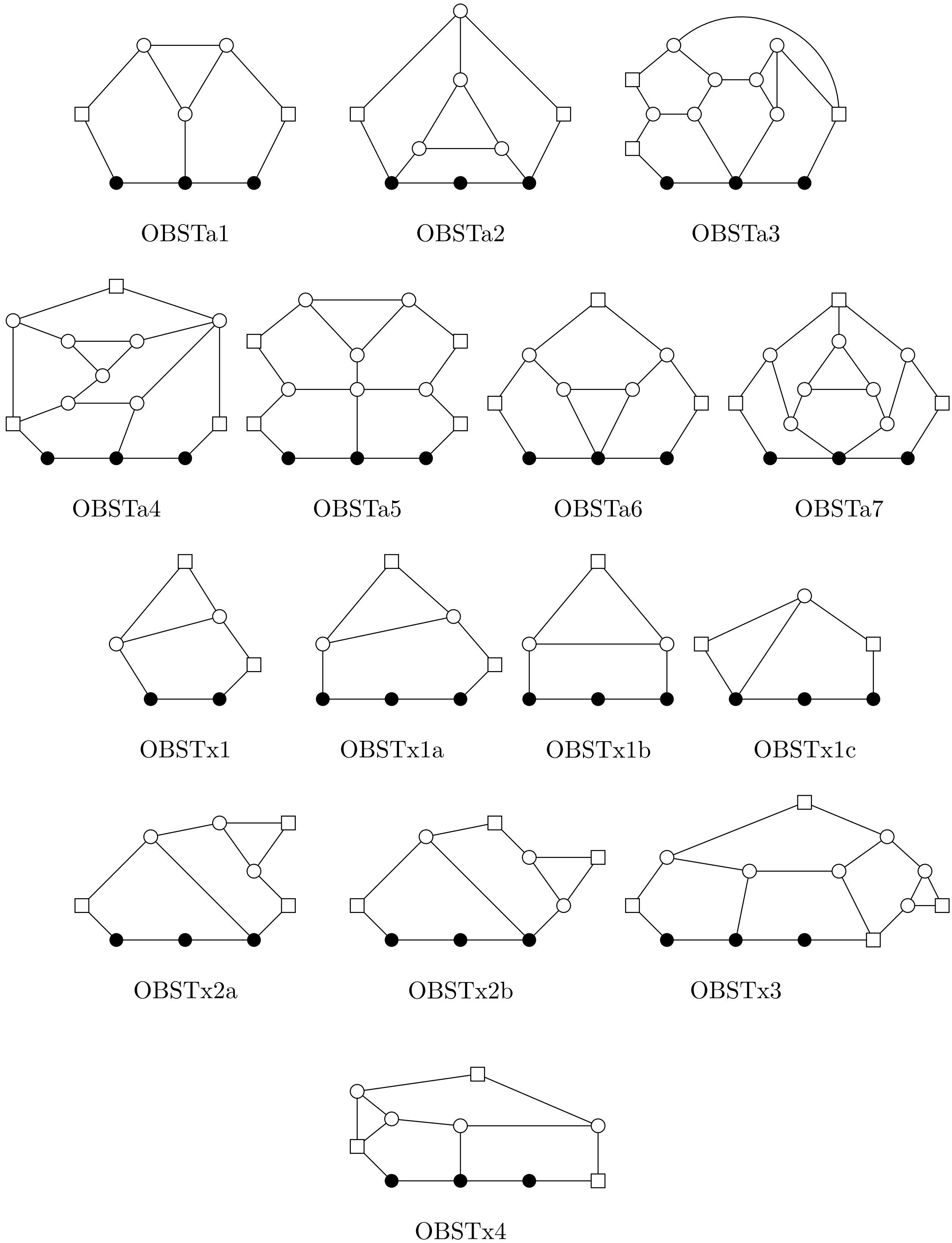


Figure 1: Forbidden configurations of Theorem 2, $\ell(P) \leq 2$

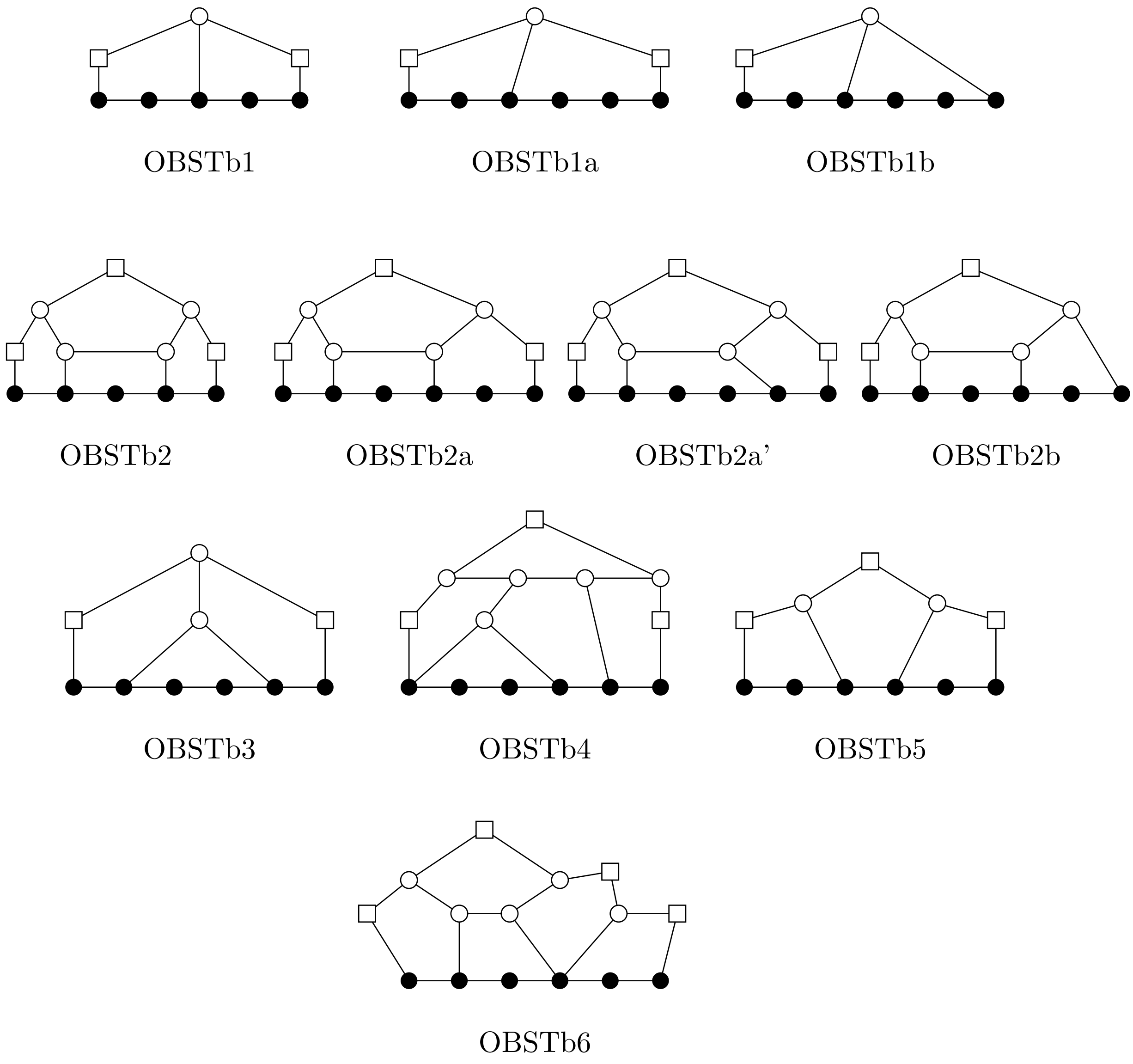


Figure 2: Forbidden configurations of Theorem 2, $\ell(P) \leq 5$

(S2) $2 \leq |L(v)| \leq 3$ *for all* $v \in V(C) \setminus V(P)$*;*

(S3) $|L(v)| = 1$ *for all* $v \in V(P)$*, and the colors in the lists give a proper coloring of the subgraph of* G *induced by* $V(P)$*;*

(I) the vertices with lists of size two form an independent set;

(T) if uvw *is a triangle,* $|L(u)| = 2$ *and* v *has a neighbor with list of size two distinct from* u*, then* w *has no neighbor with list of size two distinct from* u*; and*

(Q) if a vertex v *with list of size two has two neighbors* w_1 *and* w_2 *in* P*, then* $L(v) \neq L(w_1) \cup L(w_2)$*.*

In this situation, if $\ell(P) \leq 2$ *and*

(OBSTa) every subgraph $H \subseteq G$ *isomorphic to one of the graphs drawn in Figure 1 is* L*-colorable,*

then G *is* L*-colorable. Furthermore, if* $\ell(P) \leq 5$*,* $d(P) \geq r(P)$ *and*

(OBSTb) every subgraph $H \subseteq G$ *isomorphic to one of the graphs drawn in Figure 2 is* L*-colorable,*

then G *is* L*-colorable.*

Note that we view the single-element lists as a precoloring of the vertices of P. Also, P does not have to be a part of the facial walk of C, as we only require $V(P) \subseteq V(C)$. The notation used in Figures 1 and 2 is the following: We mark the vertices of P (precolored vertices) by full circles, the vertices with list of size three by empty circles, and the vertices with list of size two by empty squares. In the conditions (OBSTa) and (OBSTb), we require the lists of the vertices of H according to L to match the sizes prescribed by Figures 1 and 2.

Let us remark that the assumption (T) is necessary—Figure 3 shows a non-L-colorable graph G_1 with only one precolored vertex x_1 satisfying all other assumptions of Theorem 2. By repeating the left part of this graph, x_1 can be made arbitrarily far apart from the triangle. Let G_2 and G_3 with precolored vertices x_1 and x_2 be the copies of G_1 with the color A replaced by colors A' and A'', respectively, in the lists of all vertices. Let G be the graph obtained from G_1, G_2 and G_3 by identifying the vertices x_1, x_2 and x_3 to a single vertex whose list is $\{A, A', A''\}$. Note that G is a counterexample to Theorem 2 without the assumption (T) and that G has no precolored vertices and $t(G)$ can be arbitrarily large.

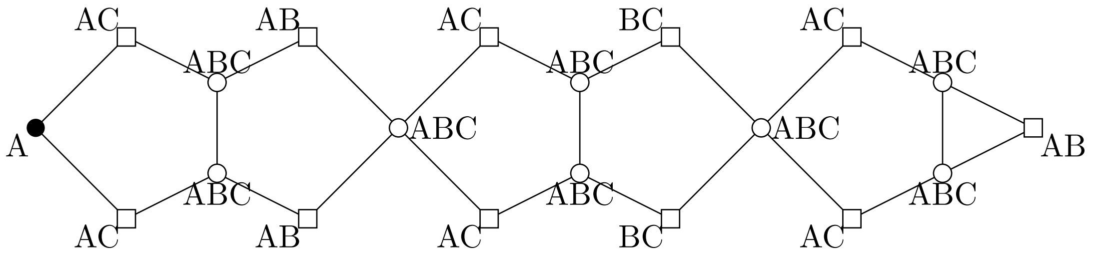
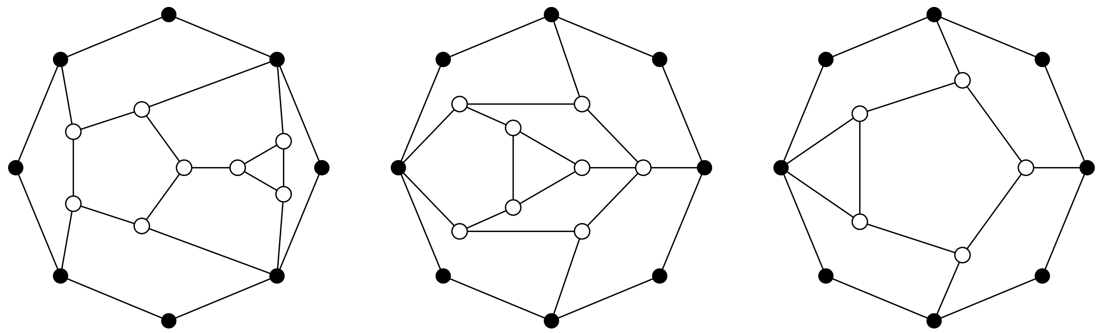


Figure 3: Assumption (T) is necessary



(a) (b) (c)

Figure 4: C-obstacles

In his paper showing that every planar graph with at most three triangles is 3-colorable, Aksionov [1] also proved that if G is a plane graph with exactly one (≤ 4)-cycle, then any precoloring of a 5-face of G extends to a 3-coloring of G. Thomassen [11] showed that in a planar graph of girth 5, any precoloring of an induced cycle C of length at most 9 extends to a 3-coloring, unless a vertex has three neighbors in C. Walls [16] extended this characterization for cycles of length at most 11 (giving more subgraphs that prevent the coloring from being extended), Thomassen [12] generalized it for list-coloring, and Dvořák and Kawarabayashi [5] extended both of these results for the cycles of length 12. Similarly, Theorem 2 implies a result regarding extension of a precoloring of a (≤ 8)-cycle, assuming that (≤ 4)-cycles are far apart.

Let C be a (≤ 8)-cycle. We call a plane graph F a *C-obstacle* if $C \subseteq F$ bounds the outer face of F, F contains exactly one (≤ 4)-cycle, and

- O1: $F - V(C)$ is a tree (with at most $\ell(C) - 6$ vertices), or
- O2: $F - V(C)$ is a graph (with at most $\ell(C) - 3$ vertices) whose only cycle is a triangle, or
- O3: F is one of the graphs drawn in Figure 4.

Corollary 3. *Let G be a plane graph with the outer face bounded by an induced (≤ 8)-cycle C, such that $t(G) \geq B$. Furthermore, assume that G*

does not contain a C-obstacle as a subgraph. Let L be an assignment of lists of size 1 to the vertices of C and lists of size 3 to the other vertices of G. If L prescribes a proper coloring of C, then G is L-colorable.

Let us give a proof of this result in a slightly more general setting, which we are going to use in the inductive proof of Theorem 2. A graph G_1 is *smaller* than G_2 if

- G_1 has smaller number of (≤ 4)-cycles than G_2, or
- G_1 and G_2 have the same number of (≤ 4)-cycles and satisfy $|V(G_1)| < |V(G_2)|$, or
- G_1 and G_2 have the same number of (≤ 4)-cycles, $|V(G_1)| = |V(G_2)|$ and $|E(G_1)| < |E(G_2)|$.

Lemma 4. *Let G be a plane graph satisfying the assumptions of Corollary 3. If Theorem 2 holds for all graphs smaller than G, then G is L-colorable.*

Proof. Suppose for a contradiction that G is a non-L-colorable graph satisfying the assumptions, such that Lemma 4 holds for all graphs smaller than G. Let $K \neq C$ be a (≤ 8)-cycle in G, and H the subgraph of G drawn in the closed disk bounded by K. If $H \neq K$, then, by the minimality of G, $G - (E(H) \setminus E(K))$ has an L-coloring φ, and since G is not L-colorable, the precoloring of K given by φ does not extend to an L-coloring of H. By the minimality of G, we conclude that either K is not an induced cycle in H or H contains a K-obstacle F. Assume the latter. Note that each internal face K' of F has length at most 7, and let H' be the subgraph of G drawn in the closed disk bounded by K'. Since F contains a (≤ 4)-cycle and $t(G) \geq B$, K' is an induced cycle in H' and H' does not contain any K'-obstacle. It follows that $H' = K'$ for every internal face K' of F, and thus $H = F$. We conclude that

every (≤ 8)-cycle $K \neq C$ in G either bounds a face, has a chord drawn inside the disk bounded by K, or the subgraph drawn inside K is a K-obstacle. (1)

In particular, every (≤ 5)-cycle bounds a face.

Consider a vertex $v \in V(G) \setminus V(C)$, and assume that v has more than one neighbor in C. If v has at least three neighbors in C, then G contains the C-obstacle consisting of v, C and three edges incident with v (satisfying the condition O1). Thus, suppose that v has exactly two neighbors $w_1, w_2 \in V(C)$. Furthermore, suppose that $\ell(C) \leq 7$ or that w_1 and w_2 are non-adjacent. Let K_1 and K_2 be the two cycles formed by w_1vw_2 and the two paths between w_1 and w_2 in C, and note that $\ell(K_1), \ell(K_2) \leq 8$ and both

K_1 and K_2 are induced cycles. By (1) and the assumption that $t(G) \geq B$, we conclude that at least one of K_1 and K_2 (say K_1) bounds a face. By the minimality of G, v has degree at least three, thus K_2 does not bound a face. Again, since $t(G) \geq B$, this implies that $\ell(K_1) \geq 5$ and $6 \leq \ell(K_2) \leq 7$. Thus, the subgraph F_2 drawn inside K_2 is a K_2-obstacle satisfying condition O1 or O2, and $F_2 \cup K_1$ is a C-obstacle in G. It follows that

no vertex $v \in V(G) \setminus V(C)$ has more than one neighbor in C, unless $\ell(C) = 8$ and the neighbors of v in C are adjacent. (2)

Also, observe that

if $\ell(C) = 8$ and v has two adjacent neighbors w_1 and w_2 in C, then no neighbor x of v distinct from w_1 and w_2 is adjacent to a vertex in C, (3)

as otherwise (1) together with $t(G) \geq B$ implies that x has two (non-adjacent) neighbors in C.

Suppose now that two adjacent vertices $v_1, v_2 \in V(G) \setminus V(C)$ both have a neighbor in C. By (2) and (3), each of them has exactly one such neighbor; let $w_i \in V(C)$ be the neighbor of v_i, for $i \in \{1,2\}$. Furthermore, suppose that both (induced) cycles K_1 and K_2 consisting of $w_1v_1v_2w_2$ together with a path joining w_1 with w_2 in C have length at least 6. Note that $\ell(K_1) + \ell(K_2) = \ell(C) + 6$, thus $\ell(K_1), \ell(K_2) \leq \ell(C)$ and $\ell(C) \geq 6$. Since $t(G) \geq B$, (1) implies that say K_1 bounds a face and the subgraph of G in K_2 is a K_2-obstacle. Consider the graph G' obtained from G by contracting an edge e of the path $K_1 - \{w_1, v_1, v_2, w_2\}$ and giving the resulting vertex a color different from the color of its neighbors. By (1), e does not belong to a (≤ 5)-cycle in G, thus the contraction does not create any (≤ 4)-cycle. Also, as G contains only one cycle of length at most 4 (drawn inside K_2), the restriction on the distance between (≤ 4)-cycles in G' is vacuously true. The graph G' is not L-colorable, and by the minimality of G, it contains an obstacle satisfying O1 or O2. However, this gives a corresponding C-obstacle in G. Therefore,

if each of two adjacent vertices $v_1, v_2 \in V(G) \setminus V(C)$ has a neighbor in C, then they together with a path in C bound a face of length at most 5. (4)

If $3 \leq \ell(C) \leq 4$, then consider the graph G' obtained from G by subdividing an edge of C by $5 - \ell(C)$ new vertices, and giving these vertices distinct colors that do not appear in any of the lists of G. Note that G' is smaller than G, since it contains fewer (≤ 4)-cycles, and by the minimality of G, we conclude that G' is L-colorable. However, that gives an L-coloring of G, thus we may assume that $\ell(C) \geq 5$.

Let us now show that there exists a set $X \subseteq V(C)$ of $\max(1, \ell(C) - 5)$ consecutive vertices of C such that

- every path of length at most 3 whose endvertices belong to X is contained in the subgraph of G induced by X, and
- no vertex of X has a neighbor in a triangle.

If $\ell(C) \leq 7$, then by (2), at most three vertices of C are incident with or have a neighbor in a triangle, and at most two vertices are incident with a 4-cycle. Since $t(G) \geq B$, these cases are mutually exclusive, thus we can choose X as a subset of the remaining (at least $\ell(C) - 3$) vertices. Hence, suppose that $\ell(C) = 8$ and $C = v_1v_2 \ldots v_8$. If say v_2v_3 is an edge of a triangle, then none of v_5, …, v_8 has a neighbor in a triangle. If $v_5v_6v_7$ is not a part of the boundary walk of a 5-face, then set $X = \{v_5, v_6, v_7\}$; otherwise, $v_6v_7v_8$ is not a part of the boundary walk of a 5-face by (2), and we set $X = \{v_6, v_7, v_8\}$. We choose the set X in the same way in case that a triangle shares a single vertex v_2 with C, or a 4-cycle shares at most two vertices v_2 and v_3 with C, or no (≤ 4)-cycle intersects C and at least 4 consecutive vertices v_5, v_6, v_7 and v_8 have no neighbor in a triangle. It remains to consider the case that no (≤ 4)-cycle intersects C and among each 4 consecutive vertices, at least one has a neighbor in a triangle. If three vertices of C had a neighbor in a triangle, then (1) would imply that $G - V(C)$ is a triangle, giving a C-obstacle satisfying O2. Therefore, two opposite vertices of C, say v_1 and v_5, have a neighbor in a triangle. However, this contradicts (2) or (4).

Let $C - X = v_1v_2 \ldots v_k$, where $k = \ell(C) - |X| \leq 5$. Let $G' = G - X$, with the list assignment L' obtained from L by removing from the list of each vertex the color of its neighbor (if any) in X. Furthermore, we set $L'(v_1) = L(v_1) \cup L(v_2)$ and $L'(v_k) = L(v_k) \cup L(v_{k-1})$. By the choice of X, G' with the list assignment L' satisfies the assumptions of Theorem 2, and every vertex incident with a triangle that does not belong to $V(C)$ has list of size three. An L'-coloring of G would correspond to an L-coloring of G, thus we conclude that $k = 5$ (and hence $\ell(C) \geq 6$) and G' contains a subgraph H isomorphic to one of the graphs OBSTa1 – OBSTa7 drawn in Figure 1 (with matching lengths of lists according to L'). However, a case analysis shows that

- if H is OBSTa1 or OBSTa2, then G contains a C-obstacle satisfying (O2),
- if H is OBSTa3, then G contains the C-obstacle drawn in Figure 4(a).
- if H is OBSTa4, OBSTa5 or OBSTa7, then G contains the C-obstacle drawn in Figure 4(b).

- if H is OBSTa6, then G contains the C-obstacle drawn in Figure 4(c).

□

Let us now give a short outline of the proof of Theorem 2. We basically follow the proof of Grötzsch theorem by Thomassen [11], which the reader should be familiar with. We consider the hypothetical smallest counterexample. First, we give constraints on short paths Q whose endvertices belong to $V(C)$ and internal vertices do not belong to $V(C)$ (claims (6), (7) and (9) in the proof), by splitting the graph along Q, coloring one part and extending the coloring to the second one, with Q playing the role of the precolored path in the second part. However, due to the existence of counterexamples to the statement "every precoloring of a path of length two can be extended" (depicted in Figure 1), we cannot exclude such paths entirely. However, using the ability to color vertices of a path of length up to 5 if we can in the process ensure that there are no ($\leq$4)-cycles nearby, we can strengthen these constraints sufficiently if the vertices of Q are close to P (claims (15) and (18)). Then, as in the Thomassen's proof, we try to color up to five appropriately chosen vertices of G near to P and remove their colors from the lists of their neighbors, so that the resulting graph G' satisfies the assumptions of Theorem 2. This may only fail if a ($\leq$4)-cycle T appears near to the colored vertices, making (I) or (T) false (claims (19) and (21)). Note that this implies that $\ell(P) \leq 2$. Many of these problematic configurations (those where T is a 4-cycle, or where (T) is false in G') can be reduced by precoloring up to three more vertices near to T, extending the precolored path and at the same time removing some vertices so that T disappears. Still, some cases (e.g., when T contains a vertex in C whose distance from P is at most four) remain. However, then we observe that we can apply the symmetric argument on the other side of P, and if that fails as well, a ($\leq$4)-cycle T' must be close to the vertices that we try to color there as well. Since the distance between any two ($\leq$4)-cycles in G is at least B, it follows that $T' = T$, which implies that G contains a short path Q with endvertices in C. Using the constraints on such paths, we can find a suitable set of vertices to color and remove in this case as well, finally finishing the proof.

Let us now provide the details of this argument, which unfortunately turns out to be rather lengthy and technical.

Proof of Theorem 2. Suppose that G together with lists L is a smallest counterexample, i.e., Theorem 2 holds for every graph smaller than G and G satisfies the assumptions of Theorem 2, but G is not L-colorable. Let C be the outer face of G and P a path with $V(P) \subseteq V(C)$ as in the statement of the theorem. We first derive several properties of this counterexample. Note that each vertex v of G has degree at least $\max(2, |L(v)|)$, and if two vertices u and v are adjacent, then $L(u) \cap L(v) \neq \emptyset$, unless uv is an edge of P. In particular, if $v \notin V(P)$ is adjacent to a vertex $p \in V(P)$, then $L(p) \subset L(v)$.

Lemma 4 implies that

every (≤ 8)*-cycle* K *in* G *either bounds a face, has a chord drawn inside the disk bounded by* K*, or the subgraph drawn inside* K *is a* K*-obstacle.* (5)

In particular, every (≤ 5)-cycle in G bounds a face. Furthermore,

The graph G *is* 2*-connected.* (6)

Proof. Clearly, G is connected. Suppose that G is not 2-connected, and let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{v\}$ and $|V(G_1)|, |V(G_2)| \geq 2$. If say $P \subseteq G_1$, then by the minimality of G, an L-coloring φ_1 of G_1 exists. Let L_2 be the list assignment such that $L_2(x) = L(x)$ for $x \neq v$ and $L(v) = \{\varphi_1(v)\}$. By the minimality of G, we have that G_2 is L_2-colorable. However, this gives an L-coloring of G. Similarly, in case that the cut-vertex v is an internal vertex of P, the minimality of G implies that both G_1 and G_2 are L-colorable, giving an L-coloring of G. This is a contradiction. □

A *chord* of a cycle K is an edge $e \notin E(K)$ joining two vertices of K. A vertex of a path is *internal* if its degree in the path is two, and an *endvertex* otherwise.

Every chord of C *joins two vertices* u *and* v *with list of size three, such that either* u *and* v *have a common neighbor with list of size two, or there exists a triangle* $w_1w_2w_3$ *with* $|L(w_2)| = 2$*, a neighbor* $z \notin \{w_2, w_3\}$ *of* w_1 *with* $|L(z)| = 2$*, and* $uz, vw_3 \in E(G)$ *or* $uw_3, vz \in E(G)$*.* (7)

Proof. Let uv be a chord of C. Let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{u, v\}$ and $|V(G_1)|, |V(G_2)| \geq 3$. By symmetry, we may assume that $|V(G_1) \cap V(P)| \geq |V(G_2) \cap V(P)|$. If $u, v \in V(P)$, then by the minimality of G, both G_1 and G_2 are L-colorable, and their colorings combine to an L-coloring of G. This is a contradiction, thus we may assume that $v \notin V(P)$. Let $P_i = (P \cap G_i) \cup \{uv\}$ for $i \in \{1, 2\}$.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment such that $L'(x) = L(x)$ for $x \notin \{u, v\}$ and $L'(x) = \{\varphi(x)\}$ for $x \in \{u, v\}$. Since G is not L-colorable, G_2 is not L'-colorable, thus it violates (Q), (OBSTa) or (OBSTb).

Suppose first that u is not an internal vertex of P. Then only two vertices are precolored in G_2, and thus G_2 contains either a vertex with list of size two adjacent to u and v or OBSTx1. By (I) and (T), neither u nor v have a list of size two. Furthermore, note that u cannot be an endvertex of P: Otherwise, we have $d(P) \leq 2$, thus $\ell(P) \leq 2$. Let $c \neq \varphi(v)$ be a color in $L(v) \setminus L(u)$ and L_2 the list assignment such that $L_2(v) = \{c\}$ and $L_2(x) = L(x)$ for $x \neq v$. Note that G_2 with list assignment L_2 satisfies (Q) and (OBSTa), and by the minimality of G, G_2 is L_2-colorable. It follows that G_1 cannot be L_2-colorable. However, we have $d(P_1) \geq B - 4 \geq r(P_1)$ in G_1. Since G_1 is not L_2-colorable, it follows that G_1 violates (Q). However, that implies that G contains a non-L-colorable OBSTx1c, OBSTx2a or OBSTx2b, which is a contradiction. Therefore, the chord uv satisfies the conclusion of (7) in this case.

Let us now consider the case that u is an internal vertex of P. By the choice of G_1 and G_2, we have $2\ell(P_2) \leq \ell(P)+2$. Suppose first that $\ell(P_2) = 2$. By the minimality of G, we conclude that (S3), (Q) or (OBSTa) fails for G_2 with the list assignment L'. This implies that $d(P) \leq 3$, and since G satisfies the assumptions of Theorem 2, we have $\ell(P) = 2$. However, by symmetry G_1 with the precolored path P_1 also fails (S3), (Q) or (OBSTa), implying that $t(G) \leq 6$. This is a contradiction.

Therefore, we may assume that $\ell(P_2) = 3$, and thus $\ell(P) \geq 4$ and $d(P) \geq r(P)$. Note that $d(P_2) \geq d(P)-1$, and thus $d(P_2) \geq r(P_2)$. By the minimality of G, we have that G_2 fails (Q), and G_2 contains a vertex w with $|L(w)| = 2$ adjacent both to v and to an endvertex of P. Analogously, G_1 (with the precolored path P_1) also fails (Q), or $\ell(P) = 5$ and G_1 fails either (S3) or (OBSTb) due to a subgraph isomorphic to OBSTb1 or OBSTb2. The obstruction in G_1 together with the 5-cycle G_2 form one of the subgraphs H described in (OBSTb), namely OBSTb1, OBSTb1a, OBSTb1b, OBSTb5 or OBSTb6; and by (5), this subgraph H is unique. By (OBSTb), H has an L-coloring ψ. However, by the minimality of G, this implies that the precoloring that assigns v the color $\psi(v)$ extends both to G_1 and G_2, contradicting the assumption that G is not L-colorable. □

Let us note that (7) implies that P is a subpath of C. Furthermore, observe that there exists an L-coloring of the subgraph of G induced by $V(C)$, unless G contains a non-L-colorable OBSTx1, OBSTx1a or OBSTx1b. Lemma 4 then implies that

$$\ell(C) \geq 9. \tag{8}$$

Proof. If $\ell(C) \leq 8$, then G would contain a C-obstacle H, and by (5), it would actually be equal to this C-obstacle. Since each C-obstacle contains a (≤ 4)-cycle whose distance from any vertex of C is at most 4, this is only possible if $\ell(P) \leq 2$. However, a straightforward case analysis shows that either G is L-colorable or violates (OBSTa). More precisely,

- If H satisfies (O1) and $|V(H) \setminus V(C)| = 1$, then G contains OBSTa1 or is L-colorable.
- If H satisfies (O1) and $|V(H) \setminus V(C)| = 2$, then G contains OBSTa6 or OBSTx4, or is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 3$, then G contains OBSTa2 or is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 4$, then G is L-colorable.
- If H satisfies (O2) and $|V(H) \setminus V(C)| = 5$, then G contains OBSTa3, OBSTa4 or OBSTa7, or is L-colorable.
- If H satisfies (O3), then G is L-colorable.

□

For $k \geq 2$, a *k-chord* of a cycle K is a path $Q = q_0q_1 \dots q_k$ of length k joining two distinct vertices of K, such that $V(K) \cap V(Q) = \{q_0, q_k\}$. We consider a chord to be a *1-chord*. Suppose that neither q_0 nor q_k is an internal vertex of P. Let G_1 and G_2 be the maximal connected subgraphs of G intersecting in Q, such that $P \subseteq G_1$. We say that Q *splits off a face* if G_2 is a cycle. For one of the obstructions O drawn in Figures 1 and 2, the k-chord Q *splits off* O if G_2 is isomorphic to O and

- the vertices drawn in the Figures by full circles coincide with the (not necessarily proper) subpath of Q consisting of the vertices $x \in V(Q)$ such that $|L(x)| \in \{1, 3\}$, and
- the sizes of the lists of all other vertices of G_2 are equal to those given by Figure 1 or 2.

Let $Q = q_0q_1 \dots q_k$ be a k-chord of C such that no endvertex of Q is an internal vertex of P and Q does not split off a face. If $k \leq 2$, or if $k = 3$ and q_3 has list of size two, then Q splits off one of the obstructions drawn in Figure 1.

(9)

Proof. Suppose for a contradiction that there exists a k-chord Q violating (9). Let G_1 and G_2 be the maximal connected subgraphs of G intersecting in Q, such that $P \subseteq G_1$. Let us choose Q among all ($\leq$3)-chords of C that violate (9) so that $|V(G_2)|$ is minimal.

By the minimality of G, there exists an L-coloring φ of G_1. Let L' be the list assignment such that $L'(x) = L(x)$ if $x \notin V(Q)$, $L'(q_3) = \{\varphi(q_2), \varphi(q_3)\}$ if $k = 3$ and $L'(q_i) = \{\varphi(q_i)\}$ for $0 \leq i \leq 2$. Observe that G_2 is not L'-colorable, thus it violates (Q) or (OBSTa). Let H be the minimal subgraph of G_2 that contains Q and violates (Q) or (OBSTa). Note that H contains a ($\leq$4)-cycle T whose distance to any vertex of H is at most four. By (5), each face of H except for the outer one is also a face of G.

We claim that $G_2 = H$, that is, Q splits off H. Otherwise, consider a k'-chord $Q' \neq Q$ of G_2 that is a subpath of the union of Q and of the outer face of H. If Q' satisfies the assumptions of (9), then by the choice of Q, we have that that Q' splits off a subgraph H' that is either a face or an obstruction drawn in Figure 1. However, H' contains a ($\leq$4)-face T', whose distance to Q' is at most three. It follows that $d(T, T') \leq 7 < B$, which is a contradiction. Therefore, Q' does not satisfy the assumptions of (9). Since every vertex with list of size two in H belongs to the outer face of G, the inspection of the graphs in Figure 1 shows that this is only possible if $k = 3$, H is OBSTx1 and $Q' = q_3q_2q_1uv$ for vertices $u, v \in V(H) \setminus V(Q)$ such that $|L(u)| = 3$ and $|L(v)| = 2$. However, in this case let G'_1 and G'_2 be the subgraphs of G that intersect in Q', let φ' be an L-coloring of G'_1 and let L_2 be the list assignment such that $L_2(x) = \{\varphi'(x)\}$ for $x \in \{v, q_1, q_2\}$, $L_2(q_3) = \{\varphi'(q_2), \varphi'(q_2)\}$, $L_2(v) = \{\varphi'(u), \varphi'(v)\}$ and $L_2(x) = L(x)$ for other vertices $x \in V(G'_2)$. Since $t(G) \geq B$ and H contains T, we conclude that G'_2 satisfies the assumptions of Theorem 2, hence G'_2 is L_2-colorable. This gives an L-coloring of G, which is a contradiction. □

(5) and (9) imply that G does not contain a subgraph isomorphic to ones described in (OBSTa) or (OBSTb), such that the sizes of the lists match those prescribed by Figures 1 and 2: If G contained such a subgraph H, we would conclude that $G = H$ as in the proof of (9), and by the assumptions, G would be L-colorable.

If $Q = q_0q_1q_2$ is a 2-chord of C in G, then at most one endvertex of Q belongs to P.

(10)

Proof. Suppose that both q_0 and q_2 belong to P. Then Q together with a subpath of P forms a cycle K of length at most $\ell(P)+2$, and by (5) together with the assumption that $d(P) \geq r(P)$ if $\ell(P) > 2$, this cycle bounds a face. Observe that q_1 cannot have a neighbor in P distinct from q_0 and q_2. Let L' be the list assignment such that $L'(q_1) \subseteq L(q_1) \setminus (L(q_0) \cup L(q_2))$ has size one and $L'(x) = L(x)$ for $x \neq q_1$. Let $G' = G - q_0q_2$ if K is a triangle and $G' = G-(V(K) \setminus V(Q))$ otherwise. Note that the vertices with list of size one form an induced path P' in G', and the length of P' is at most $\ell(P)-1$ if K has length at least 5 and at most $\ell(P)+1$ otherwise. In the former case, if $d(P) \geq r(P)$, then $d(P') \geq r(P')$, since $d(P') \geq d(P) - 1$. In the latter case, we have $\ell(P) \leq 2$ and $d(P') \geq r(P')$, since $d(K) \geq B$. Since G' is smaller than G and is not L'-colorable, we conclude that it violates (Q) or (OBSTb). However, in these cases, G itself would violate (OBSTb): If G' violates (Q), then G contains OBSTb1b; if G' contains OBSTb1, then G contains OBSTb3; and if G' contains OBSTb2, then G' contains OBSTb4. □

Suppose that C has either a 3-chord $Q = q_0q_1q_2q_3$, or a 4-chord $Q = q_0q_1q_2q_3q_4$ such that $|L(q_4)| = 2$, where no endvertex of Q is an internal vertex of P. Let G_1 and G_2 be the maximal connected subgraphs of G that intersect in Q, such that $P \subseteq G_1$. Assume that either

- *$\ell(P) \geq 4$ and $d(P, q_i) \leq r(4) - r(3) = 4$ for $0 \leq i \leq 3$, or*
- *G_1 contains a (≤ 4)-cycle T such that $d(P, q_i) \leq B - r(3)$ for $0 \leq i \leq 3$.*

Then G_2 is a 5-cycle, and hence q_0 and q_3 have a common neighbor with list of size two (equal to q_4 if Q is a 4-chord).

(11)

Proof. Let φ be an L-coloring of G_1 that exists by the minimality of G. Let L_2 be the list assignment such that $L_2(q_i) = \{\varphi(q_i)\}$ for $0 \leq i \leq 3$, if Q is a 4-chord, then $L_2(q_4) = \{\varphi(q_3), \varphi(q_4)\}$, and $L_2(x) = L(x)$ for $x \notin V(Q)$. The graph G_2 is not L_2-colorable. Furthermore, we have $d(q_0q_1q_2q_3) \geq r(q_0q_1q_2q_3)$, since either $\ell(P) \geq 4$ and $d(q_0q_1q_2q_3) + (r(4) - r(3)) \geq d(P) \geq r(P)$, or $d(q_0q_1q_2q_3) + (B - r(3)) \geq B$. By the minimality of G, we conclude that G_2 violates (Q), hence a vertex x with a list of size two is adjacent to both q_0 and q_3. Furthermore, by (5) and (9), G_2 is equal to the 5-face $q_0q_1q_2q_3x$. □

We may assume that $\ell(P) \geq 2$; otherwise, we can color $2 - \ell(P)$ vertices adjacent to P in C so that the resulting list assignment L' either still satisfies the assumptions of Theorem 2 or violates (OBSTa). But, in the latter case, (5) and (9) would imply that G with the list assignment L' is equal to one of the obstructions in Figure 1. However, then it is easy to see that G either is L-colorable or contains OBSTx1. Let $P = p_0p_1 \dots p_m$, where $m = \ell(P)$.

A subgraph H of G is a *near-obstruction* if it is isomorphic to one of the graphs in Figure 1 or 2, where the vertices drawn by full circles coincide with the vertices of H belonging to P and the sizes of lists of other vertices of H are greater or equal to the sizes prescribed by the Figure. A near-obstruction H is *tame* when for every vertex v of H that is depicted in Figure 1 or 2 by a square, if v is adjacent to a vertex in P, then $v \in V(C)$.

The graph G contains no tame near-obstruction.

(12)

Proof. Suppose that H is a tame near-obstruction in G, and let K be the cycle bounding the outer face of H. Let $Q_0 = q_0q_1 \dots q_k$ be the subpath of K vertex-disjoint with P such that $V(K) \subseteq V(Q_0) \cup V(P)$. Suppose first that both q_0 and q_k are adjacent to an endvertex of P, say q_0 to p_0 and q_k to p_m; by the assumption that $d(P) \geq r(P)$ and that H is tame and by (7), this is the case unless H is OBSTx1 and $\ell(P) = 2$. Let Q be the path consisting of Q_0 and those of the edges q_0p_0 and q_kp_m that do not belong to C.

Note that $|V(H)| < |V(G)|$, since otherwise either G violates (OBSTa) or (OBSTb), or is L-colorable. Let $G' = G - (V(H) \setminus V(Q))$. By the minimality of G, the graph H is L-colorable. Let φ be an L-coloring of H, and let L' be the list assignment such that $L'(x) = \{\varphi(x)\}$ if $x \in V(Q)$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and by the minimality of G, it cannot satisfy the assumptions of Theorem 2. But, clearly G' satisfies (I) and (T). Let us now discuss several cases; we always assume that the precolored vertices of the drawing of H in Figure 1 or 2 are labeled from left to right, i.e., p_0 is the the leftmost precolored vertex in the drawing.

- H **is OBSTx2a or OBSTx2b:** Since q_1p_2 is not a chord by (7), we have $q_1 \notin V(C)$. By (9), the 2-chord $q_0q_1p_2$ splits off a subgraph H' which is isomorphic to one of the graphs drawn in Figure 1. Since $V(H) \neq V(G)$, H' is not OBSTx1. Since $H \subseteq G$, we have that q_1 has degree at least three in H' and that q_1, p_2 and two vertices of a triangle are incident with a common 5-face in H'. This implies that H' is OBSTa1, OBSTa3 or OBSTx4. However, then q_0 is adjacent to a vertex with list of size two in H', and thus $|L(q_0)| = 3$. It follows that the 5-cycle $p_0p_1p_2q_1q_0$ has at least two L-colorings, and at least one of them extends to H'. Therefore, G is L-colorable, which is a contradiction.

- $\ell(\mathbf{Q}) \leq \mathbf{5}$**:** Since $t(G) \geq B$ or $d(P) \geq r(P)$, no vertex of Q is contained in a (≤ 4)-cycle. The inspection of the graphs depicted in Figures 1 and 2 shows that among any three consecutive internal vertices of Q, at least one has degree two in H. This implies that Q is an induced path in G, since otherwise by (5), G would contain a vertex of degree two with list of size three. Similarly, we conclude that in G, no vertex with list of size two has two neighbors in Q, unless H is OBSTa1 (or OBSTx2a, but that was already excluded). However, if H is OBSTa1 and q_0 and q_3 have a common neighbor x with list of size two, then (5) and (9) imply that $V(G) = V(H) \cup \{x\}$, and it is easy to see that G is L-colorable. We conclude that G' satisfies (S3) and (Q).

Let us discuss several subcases regarding m:

- $m = 2$: That is, H is one of the obstructions drawn in Figure 1, except for OBSTa5, OBSTx1, OBSTx2b or OBSTx3 (or OBSTx2a, which was already excluded). Note that in all these cases, $\ell(Q) \leq 4$. Also, H contains a triangle whose distance from any vertex of Q is at most three, and thus G' satisfies $d(Q) \geq r(Q)$. It follows that G' violates (OBSTb), i.e., $\ell(Q) = 4$, H is OBSTa3, OBSTa4, OBSTa6, OBSTa7, OBSTx1a, OBSTx1b or OBSTx4 and G' is OBSTb1 or OBSTb2. Since G does not contain a vertex of degree two with list of size three, if G' is OBSTb2, then H is OBSTa7. The case analysis of the possible combinations of H and G' shows that G is L-colorable, which is a contradiction.
- $m = 4$: The case that H is OBSTb1 is excluded by (9), since $d(P) \geq d(T)$, thus H is OBSTb2. (9) furthermore implies that $|L(q_2)| = 3$, and thus we may choose the L-coloring φ so that $\varphi(q_1) \notin L(q_0) \setminus L(p_0)$. Let L'' be the list assignment defined by $L''(q_0) = (L(q_0) \setminus L(p_0)) \cup \{\varphi(q_1)\}$ and $L''(x) = L'(x)$ otherwise. Note that only a path $q_1q_2q_3q_4$ of length three is precolored in G' according to this list assignment and $d(q_1q_2q_3q_4) \geq d(P) - 3 \geq r(P) - 3 \geq r(q_1q_2q_3q_4)$ and thus G' is L''-colorable. This gives an L-coloring of G, which is a contradiction.
- $m = 5$: By (10), H cannot be OBSTb3 or OBSTb4. Thus, H is OBSTb1a, OBSTb1b, OBSTb2a, OBSTb2a', OBSTb2b or OBSTb5, and $\ell(Q) \leq 4$. We conclude that G' is OBSTb1 or OBSTb2 and $\ell(Q) = 4$ (excluding the cases that H is OBSTb1a or OBSTb1b). Note that q_2 has degree two in H, and since it has degree at least three in G, we conclude that G' is OBSTb1. The case analysis of the possible combinations of H and G' shows that G is L-colorable, which is a contradiction.

- $\ell(\mathbf{Q}) > \mathbf{5}$**:** Thus, H is OBSTa5, OBSTx3 or OBSTb6. Let us discuss these cases separately:
  - H is OBSTa5: Let w be the common neighbor of q_1 and q_6, and w' the common neighbor of w, q_3 and q_4. If there exist colors $c_1 \in L(q_1) \setminus (L(q_0) \setminus L(p_0))$ and $c_2 \in L(q_6) \setminus (L(q_7) \setminus L(p_2))$ so that $L(w) \neq L(p_1) \cup \{c_1, c_2\}$, then consider the graph $G_1 = G - V(P)$ with the list assignment L_1 such that $L_1(q_1) = \{c_1\}$, $L_1(q_6) = \{c_2\}$, $L_1(w)$ chosen as an arbitrary one-element subset of $L(w) \setminus (L(p_1) \cup \{c_1, c_2\})$, $L_1(q_0) = (L(q_0) \setminus L(p_0)) \cup \{c_1\}$, $L_1(q_7) = (L(q_7) \setminus L(p_2)) \cup \{c_2\}$ and $L_1(x) = L(x)$ otherwise. The graph G_1 cannot be L_1-colorable, thus it violates (OBSTa). This is only possible if G_1 is OBSTa1, but then $V(G) = V(H)$ and thus G is L-colorable. So, we have $|L(q_0)| = |L(q_7)| = 3$, $L(q_1) = (L(q_0) \setminus L(p_0)) \cup \{c_1\}$, $L(q_6) = (L(q_7) \setminus L(p_2)) \cup \{c_2\}$ and $L(w) = L(p_1) \cup \{c_1, c_2\}$. Let ψ be an L-coloring of $q_1q_0p_0p_1p_2q_7q_6$ such that $\psi(q_1), \psi(q_6) \notin L(w) \setminus L(p_1)$. Let $G_2 = G - (V(P) \cup \{w'\})$, with the list assignment L_2 such that $L_2(x) = \{\psi(x)\}$ for $x \in \{q_0, q_1, q_6, q_7\}$, $L_2(w)$ is an arbitrary singleton list disjoint with $L_2(q_1)$ and $L_2(q_6)$ and $L_2(x) = L(x)$ otherwise. Since an L_2-coloring of G_2 corresponds to an L-coloring of G (choosing the color of w' different from the colors of q_3 and q_4, and the color of w different from the color of p_1 and w_2), we have that G_2 is not L_2-colorable. By (5), G_2 satisfies (S3) and (Q), and the internal face of G_2 incident with w has length at least six, thus G_2 satisfies (OBSTb). Furthermore, since $d(q_3q_4w') \geq B$ in G, we have $d(q_0q_1wq_6q_7) \geq B-3 \geq r(q_0q_1wq_6q_7)$. Therefore, G_2 is a counterexample to Theorem 2 smaller than G, which is a contradiction.
  - H is OBSTx3: Let $q_1w_1w_2q_3$ be the path in H such that $w_1, w_2 \neq q_2$. If $|L(q_0)| = 2$, then consider an L-coloring ψ of the subgraph of G induced by $\{q_0, q_1, w_1, w_2, p_0, p_1\}$ such that $\psi(w_2) \notin L(q_7) \setminus L(p_2)$. Let L' be the list assignment defined by $L'(q_0) = \{\psi(q_0), \psi(q_1)\}$, $L'(x) = \{\psi(x)\}$ for $x \in \{q_1, w_1, w_2\}$, $L'(q_7) = (L(q_7) \setminus L(p_2)) \cup \{\psi(w_2)\}$ and $L'(x) = L(x)$ otherwise. We conclude that $G - V(P)$ is not L'-colorable, thus it violates (OBSTa). Note that w_1 has degree two in $G - V(P)$ and the face with that it is incident does not share any vertex with the triangle, and that q_7 is not incident with the triangle, thus $G - V(P)$ contains OBSTx2a. By (5) and (9), $G - V(P)$ is equal to OBSTx2a. However, then q_2, q_5 and q_7 have list of size two and G contains OBSTx3, which is a contradiction.

So, we have $|L(q_0)| = 3$. Then, there exist $c_1 \in L(q_1) \setminus (L(w_1) \setminus L(p_1))$ and $c_0 \in L(q_0) \setminus L(p_0)$ such that $c_0 \neq c_1$. Let G_1 be the graph obtained from $G - \{p_0, p_1, w_1, w_2\}$ by adding the edge q_1p_2. Let c be a color that does not appear in any of the lists of G. Let L_1 be the list assignment such that $L_1(q_0) = \{c_0\}$, $L_1(q_1) = \{c_1\}$, $L_1(p_2) = \{c\}$, $L_1(q_7) = (L(q_7) \setminus L(p_2)) \cup \{c\}$ and $L_1(x) = L(x)$ for all other vertices of G_1. Observe that G_1 is not L_1-colorable. Furthermore, the distance of q_1 from the triangle $q_4q_5q_6$ is three both in G and G_1, and the distance of q_1 and q_7 to any other (≤ 4)-cycle is at least $B - 3$, thus $t(G_1) \geq B$. The internal face F of G_1 incident with q_1p_2 has length at least six, as otherwise the cycle $F - q_1p_2 + q_1w_1p_1p_2$ has length at most seven and contradicts (5). Furthermore, observe that neither q_0 nor q_1 is adjacent to a vertex of the triangle $q_4q_5q_6$, thus G_1 contains neither OBSTx1 nor OBSTx1a. It follows that G_1 satisfies (OBSTa), and thus it is a counterexample to Theorem 2 smaller than G. This is a contradiction.

– H is OBSTb6: Let $q_1w_1w_2p_3$ be the path in H with w_1 adjacent to p_1. If $|L(q_6)| = 2$, then let c' be the unique color in $L(q_6) \setminus L(p_5)$, and note that there exists $c \in L(q_5) \setminus (L(p_3) \cup \{c'\})$. Let $G_1 = G - \{p_4, p_5\}$ and let L_1 be the list assignment such that $L_1(q_5) = \{c\}$, $L_1(q_6) = \{c, c'\}$ and $L_1(x) = L(x)$ for $x \notin \{q_5, q_6\}$. Note that G_1 is not L_1-colorable, and since a path of length 4 is precolored in G_1 and H is a subgraph of G, we conclude that G_1 contains OBSTb2. However, this implies that G contains OBSTb6, which is a contradiction.

  Therefore, $|L(q_6)| = 3$. Then, there exists an L-coloring ψ of the subgraph of G induced by $\{q_3, q_4, q_5, q_6, p_3, p_5\}$ such that $\psi(q_3) \notin L(w_2) \setminus L(p_3)$. Let G_2 be the graph obtained from $G - (V(P) \cup \{w_1, w_2\})$ by adding a vertex w adjacent to q_0 and q_3. Let c be a new color that does not appear in $L(q_0) \cup L(q_3)$. Let L_2 be the list assignment such that $L_2(x) = \psi(x)$ for $x \in \{q_3, q_4, q_5, q_6\}$, $L_2(w) = \{c\}$, $L_2(q_0) = (L(q_0) \setminus L(p_0)) \cup \{c\}$ and $L_2(x) = L(x)$ otherwise. Observe that an L_2-coloring of G_2 corresponds to an L-coloring of G, thus G_2 is not L_2-colorable. Furthermore, a path $P_2 = wq_3q_4q_5q_6$ of length 4 is precolored in G_2. Let us remark that the newly added vertex w is not incident with a (≤ 4)-cycle, as otherwise either $t(P) < r(P)$ in G, or (5) implies that q_2 is a vertex of degree two with list of size three. Furthermore, $t(G_2) \geq B$, since only the added path q_0wq_3 could result in shortening the distance between ($\leq$4)-cycles, in G we have $d(q_0) \geq d(P) - 1 \geq r(P) - 1$ and $d(q_3) \geq d(P) - 2 \geq r(P) - 2$, and $2r(P) - 1 > B$. Also, $d(P_2) \geq d(P) - 2 \geq r(P_2)$.

Note that G_2 satisfies (S3), since w is not adjacent to q_6 and $d(P) \geq r(P)$. Similarly, G_2 satisfies (Q), since otherwise (5) would imply that q_4 is a vertex of degree two with list of size three. Hence, G_2 violates (OBSTb). Since q_4 has degree at least three, G_2 contains OBSTb1. But then q_4 and q_0 have a common neighbor x, and the existence of q_2 together with $d(P_2) \geq r(P_2)$ contradicts (5) applied to the 7-cycle $q_0q_1w_1w_2q_3q_4x$.

Finally, let us consider the case that say q_0 is not adjacent to an endvertex of P, that is, $\ell(P) = 2$, H is OBSTx1, q_0 is adjacent to p_1 and q_3 is adjacent to p_2. An L-coloring of H does not extend to an L-coloring of the subgraph G' that is split off by the path $p_0p_1q_0q_1q_2q_3$. If p_0 and q_1 have a common neighbor with list of size two, then either G is L-colorable or contains OBSTa1. Otherwise, G' satisfies (S3) and (Q), as q_1 cannot be a vertex of degree two with list of size three. Therefore, G' violates (OBSTb). If $|L(q_3)| = 2$, then G' may only be OBSTb1, OBSTb1b, OBSTb2 or OBSTb2b. OBSTb1 and OBSTb1b are excluded, since q_1 must have degree at least three; if G' is OBSTb2, then G is L-colorable, and if G' is OBSTb2b, then G contains OBSTa3. If $|L(q_3)| = 3$, then there exist L-colorings ψ_1 and ψ_2 of H such that $\psi_1(q_0) = \psi_2(q_0)$, $\psi_1(q_1) \neq \psi_2(q_1)$, $\psi_1(q_2) \neq \psi_2(q_2)$ and $\psi_1(q_3) \neq \psi_2(q_3)$. The inspection of the graphs in Figure 2 shows that at least one of ψ_1 and ψ_2 extends to an L-coloring of G', unless G' contains a subgraph H' isomorphic to OBSTb1, OBSTb1a, OBSTb1b, OBSTb3 or OBSTb5. By (5) and (9) we conclude that $G' = H$ and $G = H \cup H'$. However, all possible combinations of H and H' result in an L-colorable graph, which is a contradiction. □

Let $v_1, v_2, \ldots, v_s$ be the vertices of $C - V(P)$ labeled so that $C = p_0 \ldots p_m v_1 v_2 \ldots v_s$, where $s = \ell(C) - m - 1$. Let us also define $v_0 = p_m$.

For $1 \leq i \leq 4$, if the edge $v_{i-1}v_i$ is not contained in a cycle of length at most 4 and a vertex $v \in V(G)$ is adjacent to both v_i and an endvertex p of P, then $v \in V(C)$.

(13)

Proof. Suppose for a contradiction that $v \notin V(C)$. Let G_2 be the subgraph of G that is split off by the 2-chord v_ivp according to (9), and $G_1 = G - (V(G_2) \setminus \{v_i, v, p\})$. If $p = p_m$, then $i \in \{3, 4\}$, since $v_{i-1}v_i$ does not belong to a ($\leq$4)-cycle. By (5) and the fact that every vertex of degree two has list of size two, we have that $i = 4$ and G_2 contains a triangle. It follows that $m \leq 2$. Consider an L-coloring ψ of G_2, and let L_1 be the list assignment such that $L_1(v) = \{\psi(v)\}$, $L_1(v_4) = \{\psi(v_i)\}$ and $L_1(x) = L(x)$ for $x \notin \{v, v_4\}$. Note that G_1 is not L_1-colorable. By (7), (9), (8) and the assumption that $v \notin V(C)$, we conclude that G_1 satisfies (S3) and (Q). Therefore, using (5) and (9) we conclude that G_1 is equal to (OBSTb1) or (OBSTb2). However, all combinations of (OBSTb1) or (OBSTb2) with a $p_m v_1 v_2 v_3 v_4 v$-obstacle are L-colorable.

Let us now consider the case that $p = p_0$. Since a (≤ 4)-cycle in G_2 is in distance at most 4 from P, we have $\ell(P) \leq 2$. Let K be the cycle of length at most 8 formed by the 2-chord $v_i v p_0$, the path P, and the vertices $v_1, v_2, \ldots, v_i$. Since $t(G) \geq B$, G_1 cannot be a K-obstacle, and if K is not a face, then $\ell(K) = 8$ and K has a chord splitting K to two 5-faces. If K is not a face, then since each vertex with list of size three has degree at least three, we conclude that $|L(v_1)| = |L(v_3)| = 2$, $|L(v_2)| = 3$ and the chord of K is $v_2 p_0$. However, this contradicts (7). Therefore, K is a face. Since v has degree at least three, G_2 is not a face. Furthermore, G_2 is not (OBSTx1b), thus $|L(v_i)| = 3$. Hence, there exist L-colorings ψ_1 and ψ_2 of K such that $\psi_1(v) \neq \psi_2(v)$ and $\psi_1(v_i) \neq \psi_2(v_i)$. The inspection of the graphs in Figure 1 shows that at least one of ψ_1 and ψ_2 extends to an L-coloring of G_2, giving an L-coloring of G. This is a contradiction. □

Suppose that $m = 5$. For $1 \leq i \leq 4$, if a vertex $v \in V(G)$ is adjacent to both v_i and to $p \in \{p_1, p_4\}$, then $v \in V(C)$, unless $p = p_4$ and $i = 2$, or $p = p_1$ and $i = s - 1$.

(14)

Proof. Suppose that $v \notin V(C)$ is adjacent to p_4 and v_i. Since $d(P) \geq r(P)$ and every vertex with list of size three has degree at least three, (5) implies that $i = 2$.

Hence, assume that $v \notin V(C)$ is adjacent to p_1 and v_i. Let $Q = p_0 p_1 v v_i$, let G_1 be the subgraph of G drawn in the cycle bounded by $v p_1 \ldots p_5 v_1 \ldots v_i$ and $G_2 = G - (V(G_1) \setminus V(Q))$. By the minimality of G, there exists an L-coloring φ of G_1. Let L_2 be the list assignment such that $L_2(x) = \varphi(x)$ for $x \in \{v, v_i\}$ and $L_2(x) = L(x)$ otherwise; the graph G_2 cannot be L_2-colorable. Since only an induced path Q of length three is precolored in G_2 (and $d(Q) \geq d(P) - 2 \geq r(P) - 2 \geq r(Q)$), we conclude that G_2 violates (Q), thus there exists a vertex w with list of size two adjacent to p_0 and v_i. By (7), we have $C = p_0 p_1 \ldots p_5 v_1 \ldots v_i w$, and thus $i = s - 1$. □

If v_i has degree two and is incident with a triangle, then $i \geq 4$. Furthermore, if $4 \leq i \leq 6$, v_i has degree two and is incident with a triangle, then $|L(v_{i+2})| \neq 2$. (15)

Proof. Suppose first that v_i is such a vertex, with $1 \leq i \leq 3$. Clearly, this is only possible if $\ell(P) \leq 2$. By the minimality of G, the subgraph G_0 of G induced by $V(P) \cup \{v_1, \ldots, v_{i+1}\}$ has an L-coloring ψ. Let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, \ldots, v_{i+1}\}$ and $L'(x) = L(x)$ otherwise, and let $Q = p_0p_1p_2v_1 \ldots v_{i-1}v_{i+1}$. Let $G' = G - v_i$. Then, G' is not L'-colorable. Furthermore, by (7) and (8), G' satisfies (Q). Since $d(Q) \geq d(v_{i-1}v_iv_{i+1}) - 4 \geq B - 4 \geq r(Q)$, G' violates (OBSTb), and by (5) and (9), G' is equal to one of the graphs drawn in Figure 2. If $i = 2$, then either G' is OBSTb1 and thus G contains OBSTx2b, or G' is OBSTb2 and G is L-colorable. Therefore, $i = 3$. If $|L(v_1)| = 3$, then we can assume that $\psi(v_2) \notin L(v_1) \setminus L(p_2)$, thus there exist two L-colorings of the subgraph of G_0 that differ only in the color of v_1. Furthermore, the degree of v_1 in G' is at least three. The inspection of the graphs drawn in Figure 2 shows that at least one of these colorings extends to G', which is a contradiction. If $|L(v_1)| = 2$, then by (T) we have that either G' is OBSTb1b and G contains OBSTx2a, or G' is OBSTb2b and G contains OBSTx3.

Suppose now that $4 \leq i \leq 6$ and $|L(v_{i+2})| = 2$. Again, $m = 2$. By (T), $|L(v_{i-2}| = 3$, and by (7), $p_0p_1p_2v_1 \ldots v_{i-1}v_{i+1}$ is an induced path. Thus, there exists its L-coloring ψ such that $L(v_i) \neq \{\psi(v_{i-1}), \psi(v_{i+1})\}$ and $\psi(v_{i+1}) \notin L(v_{i+2})$. Let $G' = G - \{v_{i-1}, v_i, v_{i+1}\}$ with the list assignment L' such that $L'(v_j) = \{\psi(v_j)\}$ for $1 \leq j \leq i-3$, $L'(v_{i-2}) = \{\psi(v_{i-3}), \psi(v_{i-2})\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ for a vertex $x \in V(G')$ with a neighbor $y \in \{v_{i-1}, v_{i+1}\}$ and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. Furthermore, by (7), (S2) holds, and by (9), (I) is satisfied as well. Let w be a common neighbor of two vertices of the path $Q = p_0p_1p_2v_1 \ldots v_{i-3}$ in G'. By (7), we have $w \neq v_{i-2}$ and $|L(w)| = 3$. Furthermore, $|L'(w)| = 3$, since otherwise w would be adjacent to v_{i-1} or v_{i+1} as well, and (5) would imply that v_{i-2} has degree two in G. This shows that (Q) is true. Note that $d(Q) \geq B-7 > r(P)$. Therefore, G' violates (OBSTb). This implies that $i \geq 5$; observe that there exist L-colorings ψ_1 and ψ_2 of Q such that $\psi_1(v_{i-1}) = \psi_2(v_{i-1}) = \psi(v_{i-1})$, $\psi_1(v_{i+1}) = \psi_2(v_{i+1}) = \psi(v_{i+1})$, $\psi_1(v_{i-2}) \neq \psi_2(v_{i-2})$, $\psi_1(v_{i-3}) \neq \psi_2(v_{i-3})$ and if $i = 6$, then $\psi_1(v_1) = \psi_2(v_1)$. Note that v_{i-4} is not adjacent to a vertex x with $|L'(x)| = 2$ and that v_{i-2} is the only such vertex adjacent to v_{i-3}, by (7), (5) and the fact that v_{i-2} has degree at least three in G. Since neither ψ_1 nor ψ_2 extends to an L'-coloring of G', the inspection of the graphs depicted in Figure 2 shows that $i = 6$ and G' contains OBSTb3. If v_8 is adjacent to p_0, then G contains OBSTx3. Otherwise, (7) and (9) imply that the edge of OBSTb3 incident with v_{i-2} (distinct from $v_{i-3}v_{i-2}$) is a chord of C that splits off OBSTx1 in G; however, the resulting graph is L-colorable. □

We have $|L(v_1)| = 2$ *or* $|L(v_2)| = 2$.

(16)

Proof. Suppose that $|L(v_1)| = |L(v_2)| = 3$. Let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_m)$ and $L'(x) = L(x)$ otherwise. Let $G' = G - p_m v_1$. By (7), G' with the list assignment L' satisfies (I). Suppose that (T) is violated. Then there exists a triangle $w_1w_2w_3$ such that either $v_1 = w_2$ and both w_1 and w_3 have a neighbor with list of size two, or $|L(w_2)| = 2$, w_1 is adjacent to v_1 and w_3 has a neighbor w distinct from w_1 with list of size two. By (9), the former is not possible, and in the latter case, we have $w_1 = v_2$, $w_2 = v_3$ and $w_3 = v_4$. However, that contradicts (15). Therefore, (T) holds. Furthermore, by (7), v_1 is not adjacent to any vertex of P other than p_m, and thus (Q) is satisfied. Since an L'-coloring of G' would give an L-coloring of G, it follows that G' with the assignment L' violates (OBSTa) or (OBSTb). However, this implies that G with the list assignment L contains a tame near-obstruction H, contradicting (12). □

If $\ell(P) = 5$, *then* $\ell(C) \geq 10$.

(17)

Proof. By (8), we have $\ell(C) \geq 9$. Suppose that $\ell(C) = 9$. By (16), either $|L(v_1)| = 2$ or $|L(v_2)| = 2$. Applying (16) symmetrically on the other end of P, we also have that $|L(v_2)| = 2$ or $|L(v_3)| = 2$. Therefore, either $|L(v_1)| = |L(v_3)| = 2$ and $|L(v_2)| = 3$, or $|L(v_1)| = |L(v_3)| = 3$ and $|L(v_2)| = 2$. In the former case, L-color the path $v_1v_2v_3$ so that v_1 gets a color different from the color of p_5 and v_3 a color different from the color of p_0. Let $G' = G - \{v_1, v_2, v_3\}$, with the list assignment L' obtained from L by removing the colors of the vertices v_1, v_2 and v_3 from the lists of their neighbors. Note that G' satisfies (I), since otherwise $v_1v_2v_3$ would be a part of a 5-cycle, and by (5), v_2 would have degree two. Furthermore, (T) is satisfied since $d(P) \geq r(P)$ and (Q) is satisfied by (10). Note also that no vertex adjacent to p_0 or p_5 has list of size 2, thus G' satisfies (OBSTb). This is a contradiction, since an L'-coloring of G' corresponds to an L-coloring of G.

In the latter case, let G' be the graph with list assignment L' obtained from G by coloring v_2 from its list arbitrarily, removing v_2 and removing its color from the lists of its neighbors. Again, (I), (T) and (Q) are obviously satisfied by G'. Furthermore, since $d(P) \geq r(P)$, the distance between any pair of vertices of G' with list of size two is at least three. This implies that G' satisfies (OBSTb), unless it contains OBSTb1b. However, that is excluded by (10). □

Let X be the set of vertices defined as follows: If $|L(v_1)| = 3$ (and thus $|L(v_2)| = 2$ by (16) and $|L(v_3)| = 3$) and $|L(v_4)| = 3$, then $X = \{v_2\}$. If $|L(v_1)| = 3$ and $|L(v_4)| = 2$, then $X = \{v_2, v_3\}$. If $|L(v_1)| = 2$ (and thus $|L(v_2)| = 3$) and $|L(v_3)| = 3$, then $X = \{v_1\}$. If $|L(v_1)| = |L(v_3)| = 2$ (and thus $|L(v_4)| = 3$) and $v_5 = p_0$ or $|L(v_5)| = 3$, then $X = \{v_2, v_3\}$. Otherwise, $X = \{v_2, v_3, v_4\}$.

Let $Q = q_0q_1 \ldots q_k$ be a k-chord of C such that no endvertex of Q is an internal vertex of P and Q does not split off a face. If $k \leq 2$, or if $k = 3$ and q_3 has list of size two, then $q_0 \notin X$.

(18)

Proof. Let G_2 be the subgraph of G that is split off by Q and $G_1 = G - (V(G_2) \setminus V(Q))$. Let Q be chosen so that G_2 is as large as possible. Let i be the index such that $v_i = q_0$. By (9) we can assume that $\ell(P) = 2$, since otherwise G_2 contains a triangle whose distance from q_0 is at most four, hence its distance from P is at most 8, contradicting $d(P) \geq r(P)$.

By (7) and (15), the path consisting of P and $v_1v_2v_3v_4$ is induced. Suppose now that $q_k \in \{v_1, v_2, v_3, v_4\}$, and let K be the cycle bounded by Q and a subpath of $v_1v_2v_3v_4$. Since Q does not split off a face, (5) implies that $\ell(K) \geq 6$, thus $k = 3$ and $\{q_0, q_k\} = \{v_1, v_4\}$. If $q_0 = v_1 \in X$, then $|L(v_1)| = 2$ and $|L(v_2)| = |L(v_3)| = 3$. However, (5) implies that v_2 or v_3 has degree two, which is a contradiction.

If $q_0 = v_4 \in X$, then (5), (15) and the choice of X imply that either $v_2q_2 \in E(G)$, or v_2, q_2 and q_0 are adjacent to vertices of a triangle T. In the former case, let ψ_1 and ψ_2 be L-colorings of the subgraph of G induced by $V(P) \cup \{v_1, v_2, q_2\}$ such that $\psi_1(v_1) = \psi_2(v_1)$, $\psi_1(v_2) \neq \psi_2(v_2)$ and $\psi_1(q_2) \neq \psi_2(q_2)$, let $G' = G - v_1v_2$ and let L_1 and L_2 be the list assignments such that $L_j(x) = \{\psi_j(x)\}$ for $x \in \{v_1, v_2, q_2\}$ and $L_j(x) = L(x)$ otherwise. Note that G' satisfies (Q) by (9) and that G' is not L_j-colorable for $j \in \{1, 2\}$, thus G' with both of these assignments violates (OBSTb). This is only possible if G' contains OBSTb3, but then G contains OBSTx4. In the latter case, let t_1 and t_2 be the vertices of T adjacent to v_2 and v_4, respectively, let ψ be an

L-coloring of $p_m v_1 v_2 v_3 v_4$ such that either $\psi(v_2) \notin L(t_1)$ or $L(t_1) \setminus \{\psi(v_2)\} \neq L(t_2) \setminus \{\psi(v_4)\}$, and let G' be the graph obtained from $G - V(T)$ by identifying v_2 with v_3 to a new vertex z. Note that z is not contained in a ($\leq$4)-cycle by (5), and observe that $t(G') \geq B$. let L' be the list assignment defined in the following way: $L'(v_i) = \{\psi(v_i)\}$ for $i \in \{1, 4\}$, $L'(z) = \{c\}$ for a new color c that does not appear in any of the lists, and $L'(x) = L(x)$ for any other vertex $x \in V(G')$. Observe that G' is not L'-colorable and satisfies (Q) by (7) and (8), hence G' contains a subgraph H violating (OBSTb). Since q_1 has degree at least three, (5) implies that $v_1 z v_4 q_1 q_2$ is the only cycle of length at most 5 in G' containing z, and that every cycle of length 6 containing z also contains q_1. It follows that $q_1 \in V(H)$. Unless H is OBSTb1b or OBSTb2b, $|L'(q_1)| = 3$ implies that $v_5 \in V(H)$, thus v_4 has degree at least three in H. Note that H is neither OBSTb1b nor OBSTb2b, since then we would have $v_5 \notin V(H)$ and a ($\leq$3)-chord contained in the outer face of H incident with v_4 would contradict (9). The only obstruction in that the endvertex of the precolored path has degree greater than two is OBSTb4, however H is not OBSTb4 since q_1 is not adjacent to p_m.

Therefore, $q_k \notin \{v_1, v_2, v_3, v_4\}$. By (9), G_2 is one of the graphs depicted in Figure 1. Observe that there exists a color $c \in L(q_0)$ such that every L-coloring of Q that assigns c to q_0 extends to an L-coloring of G_2. Suppose first that there exists an L-coloring ψ of the path $P' = p_0 p_1 p_2 v_1 \ldots v_i$ such that $\psi(q_0) = c$. Let L_1 be the list assignment such that $L_1(x) = \{\psi(x)\}$ for $x \in \{v_1, \ldots, v_{i-1}\}$, $L_1(v_i) = \{\psi(v_i), \psi(v_{i-1})\}$ and $L_1(x) = L(x)$ otherwise. Note that the path $P_1 = P' - v_i$ that is precolored in G_1 has length at most 5. Furthermore, G_2 contains a triangle whose distance from v_i is at most 4, thus $d(P_1) \geq B - 10 \geq r(P_1)$, and since G is not L-colorable, G_1 is not L_1-colorable. By (7), G_1 satisfies (I) and (Q). Note that the distance in G_1 from v_i to any triangle is at least $B - 4 > 1$, thus G_1 satisfies (T). We conclude that G_1 violates (OBSTb), and thus $i \in \{3, 4\}$. The choice of Q implies that if $Q' \neq Q$ is a path in G_1 of length at most three from a vertex v_j with $j \leq i$ to a vertex with list of size two, then the endvertex of Q' is q_0 and Q' bounds a face. The inspection of the graphs in Figure 2 shows that G_1 can only satisfy this condition if it contains OBSTb1, OBSTb1a or OBSTb1b. However, if G_1 contains one of these graphs, then (5) and (7) imply that both v_1 and v_2 have degree two, which is a contradiction.

Let us now consider the case that there is no L-coloring of the path P' assigning the color c to v_i. Since the path P' is induced, this is only possible if $i = 1$, or if $i = 2$ and $|L(v_1)| = 2$. If $|L(v_i)| = 2$, then $i = 1$ and (9) implies that $k = 2$ and G_2 is OBSTx1b. However, that is excluded by (15). Therefore, $|L(v_i)| = 3$. There exist two L-colorings ψ_1 and ψ_2 of P' such that $\psi_1(v_i) \neq \psi_2(v_i)$, and by the minimality of G, both of them extend to

L-colorings φ_1 and φ_2 of G_1. Furthermore, neither φ_1 nor φ_2 extends to an L-coloring of G_2. The inspection of the graphs in Figure 1 shows that this is only possible if G_2 is OBSTa1 or OBSTx1c, or if $k = 3$ and G_2 is OBSTa2 or OBSTx2a. The case that G_2 is OBSTx2a is excluded by (15). Let us discuss the rest of the cases separately:

- If G_2 is OBSTa1, then there exists a color $c_1 \in L(q_1) \setminus \{\psi_1(q_0)\}$ such that every coloring of Q that assigns $\psi_1(q_0)$ to q_0 and c_1 to q_1 extends to an L-coloring of G_2. By (9), no neighbor of q_1 has list of size two. Let L' be the list assignment such that $L'(v_j) = \{\psi_1(v_j)\}$ for $1 \le j \le i$, $L'(q_1) = \{\psi_1(q_0), c_1\}$ and $L'(x) = L(x)$ otherwise. Note that G_1 is not L'-colorable, thus it violates (Q) or (OBSTb). If (OBSTb) is violated, i.e., G_1 contains OBSTb1 or OBSTb2, then G contains a (≤ 3)-chord contradicting the choice of Q, thus suppose that (Q) is false. Then, (9) implies that $i = 2$ and q_1 is adjacent to p_1. However, then consider the path $Q' = p_0p_1q_1q_2$ (or $Q' = p_0p_1q_1q_2q_3$ if $k = 3$). Similarly to (11), we conclude that p_0 and q_2 have a common neighbor with list of size two, and since q_2 has degree at least three, this common neighbor is not equal to q_3. However, then G contains OBSTa5.

- If G_2 is OBSTx1c, then by (15), q_0 has degree two in G_2. Since neither φ_1 nor φ_2 extends to an L-coloring of G_2, this implies that Q is a 3-chord. Note that there exists an L-coloring φ of the path $p_mv_1 \ldots v_{i+2}$ such that $\varphi(v_{i+2}) \notin L(q_3)$. Let L' be the list assignment such that $L'(v_j) = \{\varphi(v_j)\}$ for $1 \le j \le i+1$, $L'(v_{i+2}) = \{\varphi(v_{i+1}), \varphi(v_{i+2})\}$ and $L'(x) = L(x)$ otherwise. The graph $G' = G - v_{i+2}q_3$ is not L'-colorable, thus it contains a subgraph H violating (OBSTb). By (9), if $i = 2$ then G' does not contain OBSTb1 or OBSTb2, hence $v_{i+1}, v_{i+2} \in V(H)$. By (5), we conclude that v_i has degree at least three in H, and by the choice of Q, we have $q_3 \in V(H)$. By (5) and (9), we have $G' = H$. If H is OBSTb3, then G is OBSTx4. Otherwise, G contains a subgraph H' depicted in Figure 5. Observe that every L-coloring of $G - V(H')$ extends to an L-coloring of G, contradicting the minimality of G.

- If G_2 is OBSTa2, then let w_1 and w_2 be the neighbors of v_i and v_{i+2}, respectively, that are incident with the triangle T of the configuration. Since neither φ_1 nor φ_2 extends to an L-coloring of G_2, we have $L(w_1) = L(w_2)$. Let φ be a coloring of the path $p_mv_1 \ldots v_{i+2}$ such that $\varphi(v_i) \ne \varphi(v_{i+2})$. Let G' be the graph obtained from $G - (V(T) \cup \{v_{i+1}\})$ by adding the edge v_iv_{i+2}, and L' the list assignment such that $L'(v_j) = \{\varphi(v_j)\}$ for $1 \le j \le i+2$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. By (5), no (≤ 4)-cycle in G' contains the edge v_iv_{i+2},

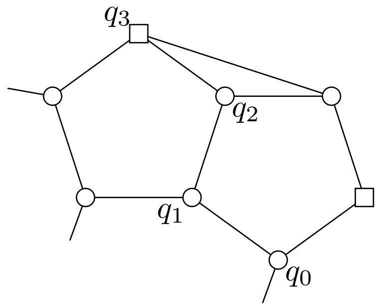


Figure 5: A configuration from claim (18).

thus the minimality of G implies that G' violates (Q) or (OBSTb). If G' violates (Q), then q_3 is adjacent to p_0, and since q_1 has degree at least three, (5) applied to the cycle $p_0p_1 \ldots q_0q_1q_2q_3$ shows that $i = 2$ and q_1 is adjacent to p_1. It follows that G contains OBSTa4. Suppose now that G' contains a subgraph H violating (OBSTb). Observe that v_{i+3} belongs to H; and, the inspection of the graphs in Figure 2 shows that v_{i+3} has degree two in H. However, since Q is a 3-chord, $v_{i+3} = q_3$ has degree at least three in G, contradicting (5) or (9).

□

Let k be the index such that $v_k \in X$ and $v_{k+1} \notin X$. We now show that G contains one of several subgraphs near to P; see Figure 6 for cases (A4) and (A5).

One of the following holds:

(A1) $|X| = 3$ *and* $v_2v_3v_4$ *is a part of the boundary walk of a* 5*-face, or*

(A2) a vertex of X *is incident with a triangle, or*

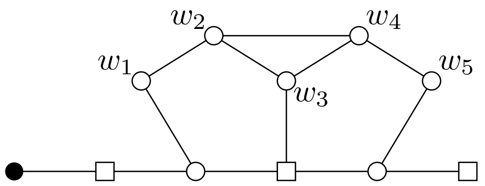
(A3) an edge of the path $p_mv_1v_2 \ldots v_k$ *is incident with a* 4*-face, or*

(A4) $|X| = 3$ *and there exists a path* $w_1w_2w_3w_4w_5$ *in* $G - (X \cup V(P))$ *such that* $w_2w_4, v_2w_1, v_3w_3, v_4w_5 \in E(G)$*, or*

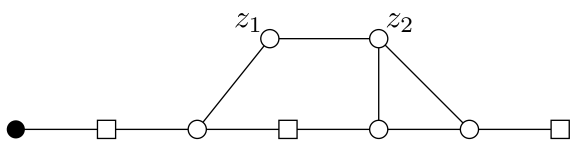
(A5) $|L(v_1)| = |L(v_3)| = |L(v_6)| = 2$ *and there exist adjacent vertices* $z_1, z_2 \in V(G) \setminus (X \cup V(P))$ *such that* $z_1v_2, z_2v_4, z_2v_5 \in E(G)$*.*

(19)

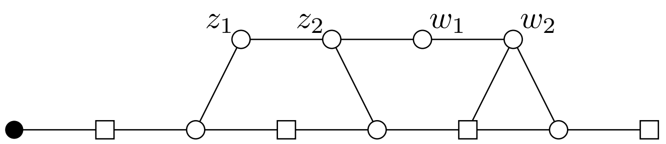
Proof. Assume for a contradiction that X satisfies none of these conditions. Since no vertex of X is incident with a triangle, (7) implies that the subgraph



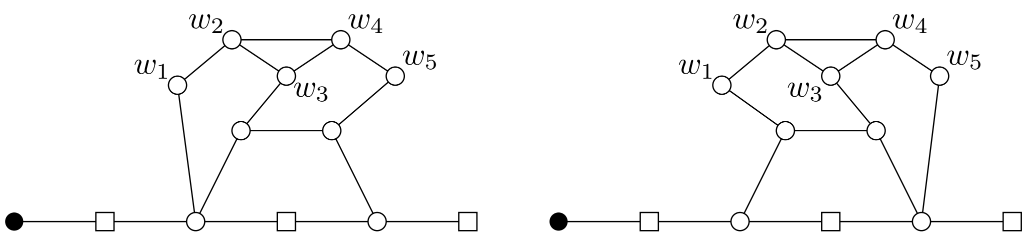
(A4)



(A5)



(B3)



(B4)

Figure 6: Configurations from claims (19) and (21)

R induced by $V(P) \cup \{v_1, \ldots, v_k\}$ is either a path or equal to the cycle C. Observe that there exists an L-coloring ψ of R such that

- if $v_1 \in X$, then $\psi(v_1) \notin L(p_m)$, and
- if $v_1 \notin X$ and $|L(v_1)| = 2$, then $\psi(v_2)$ is different from the unique color in $L(v_1) \setminus L(p_m)$, and
- if $|L(v_{k+1})| = 2$, then $\psi(v_k) \in L(v_k) \setminus L(v_{k+1})$.

Let $G' = G - X$ and let L' be the list assignment obtained from L by removing the colors of vertices of X from the lists of their neighbors, with the following exception: if $v_1 \notin X$ and $|L(v_1)| = 2$, then $L'(v_1) = L(v_1)$ (note that still, an L'-coloring of G' corresponds to an L-coloring of G, since $\psi(v_2)$ does not belong to $L(v_1) \setminus L(p_m)$). By (7), no neighbor of a vertex of X other than v_1 and v_{k+1} has list of size less than three in L; furthermore, since (A2) and (A3) are false, no vertex of G' has two neighbors in X. It follows that G' satisfies (S2). By (7) and (10), no vertex of $V(G) \setminus V(P)$ has two neighbors in P, thus (Q) holds. Let us now show that (I) holds: otherwise, there would exist adjacent vertices $w_1, w_2 \in V(G')$ such that $|L'(w_1)| = |L'(w_2)| = 2$. We may assume that $|L(w_1)| = 3$, and thus w_1 has a neighbor in X. If $|L(w_2)| = 3$, then w_2 has a neighbor in X as well, and by (5), it follows that (A1), (A2) or (A3) holds. If $|L(w_2)| = 2$ and $w_1 \notin V(C)$, then (18) is contradicted, unless (A2) holds. If $w_1 \in V(C)$, then since (A2) is false, (18) implies that $w_1 \in \{v_1, v_{k+1}\}$. If $w_1 = v_1$, then the chord w_1w_2 contradicts (7), hence $w_1 = v_{k+1}$ and $w_2 = v_{k+2}$. However, the set X was chosen so that if $|L(v_{k+1})| = 3$, then $|L(v_{k+2})| = 3$, which is a contradiction.

Suppose now that (T) is violated, that is, there exists a path $w_1w_2w_3w_4w_5$ in G' such that $|L'(w_1)| = |L'(w_3)| = |L'(w_5)| = 2$ and $w_2w_4 \in E(G)$. If $|L(w_3)| = 2$, then by (T) and symmetry, we may assume that $|L(w_1)| = 3$, and hence w_1 has a neighbor $x \in X$. If $w_1 \notin \{v_1, v_{k+1}\}$, then (18) implies that a subpath of $xw_1w_2w_3$ splits off a face F, and since $|L(w_3)| = 2$, we have $\ell(F) \leq 4$. However, $d(F, w_2w_3w_4) < B$, which is a contradiction. If $w_1 = v_1$, then by (9), a subpath of $w_1w_2w_3$ splits off a triangle T or OBSTx1. However, then (A2) holds. It follows that $w_1 = v_{k+1}$. If $|L(w_5)| = 3$, by symmetry we have $w_5 = v_{k+1} = w_1$, which is a contradiction. Therefore, $|L(w_5)| = 2$ and by (9), $w_3w_4w_5$ is a subpath of C. Since the triangle $w_2w_3w_4$ is outside of the subgraph split off by $w_1w_2w_3$, we also conclude that $w_1w_2w_3 \subset C$, thus $w_j = v_{k+j}$ for $1 \leq j \leq 5$. However, then $k \leq 3$, since both v_{k+1} and v_{k+2} have a list of size three, and $|L(v_{k+5})| = 2$ and v_{k+3} is a vertex of degree two incident with a triangle, contradicting (15). Thus, $|L(w_3)| = 3$ and w_3 has a neighbor $y \in X$. If $|L(w_1)| = |L(w_5)| = 3$, then each of them has a neighbor in X, and thus (A4) holds. Therefore, assume that say $|L(w_1)| = 2$. If $w_3 \not\in \{v_1, v_{k+1}\}$, then by (18) a subpath of $yw_3w_2w_1$ splits off a face of length at most four whose distance from $w_2w_3w_4$ is less than B, which is a contradiction. Similarly, (9) shows that $w_1w_2w_3 \subset C$, hence $w_j = v_{k+4-j}$ for $1 \leq j \leq 3$. If $|L(w_5)| = 2$, a symmetrical argument would show that $w_5 = v_{k+3} = w_1$, thus we have $|L(w_5)| = 3$ and w_5 has a neighbor in X. By the choice of X, it follows that (A5) holds.

Therefore, G' satisfies (S1), (S2), (S3), (I), (Q) and (T), and by the minimality of G, we conclude that G' violates (OBSTa) or (OBSTb). Thus G contains a near-obstruction H, and by (12), there exists a vertex $v \in V(H)\setminus V(C)$ such that $|L'(v)| = 2$. By (13), v is not adjacent to an endvertex of P, hence either $m = 2$ and H is OBSTx1, or $m = 5$ and H is OBSTb1 or OBSTb2, with a vertex $p \in \{p_0, p_m\}$ not contained in H. Let v_t be the neighbor of v in X.

Suppose first that $m = 2$. Let $q_0q_1q_2q_3$ be the subpath of the outer face of H, where $q_0q_2 \in E(G)$ and $q_3 = v$. If $p = p_0$, then H is drawn inside the closed disk bounded by $K = p_2p_1vv_tv_{t-1}\ldots v_1$. Then, (5) implies that $t \geq 3$. Since at most one of v_2 and v_3 has degree two, only the vertices q_0, q_1 and q_2 are contained in the open disk bounded by K. Since at most one of v_1 and v_2 has degree two, v_2 is adjacent to a vertex of the triangle $q_0q_1q_2$. Considering the path $Q = p_0p_1vv_t$, as in (11) we conclude that Q splits off a face and p_0 and v_t have a common neighbor with list of size two. However, such a graph G is L-colorable. Hence, suppose that $p = p_2$ and observe that $t = 2$ and v_2 has list of size three. Therefore, v_2 has degree at least three, and $q_1, q_2, q_3 \not\in V(C)$ by (11). It follows that $|L(q_1)| = 3$ and q_1 is adjacent to a vertex $x \in X$. Note that x and p_0 have a common neighbor with list of size two by (11) applied to $xq_1q_0p_0$. But, such a graph G is L-colorable.

Let us now consider the case that $m = 5$. By (14) and symmetry (we will no longer use any properties of the set X), we may assume that $p = p_5$ and v is adjacent to v_2 and p_4. Let K be the cycle bounding the outer face of H and $Q = K - (V(P) \cup \{v_1\}) = q_0q_1\ldots$, where q_0 is adjacent to p_0. By (13), we have $q_0 \in V(C)$. Let $G_1 = G - (V(H) \setminus V(Q))$.

If H is OBSTb1, then note that v_2 has degree at least three, thus by (11) q_0 and v_2 have a common neighbor with list of size two. However, then G contains OBSTb2b. Therefore, H is isomorphic to OBSTb2. There exists an L-coloring φ of H such that $\varphi(q_1) \not\in L(q_0) \setminus L(p_0)$. Let L_1 be the list assignment defined by $L_1(x) = \varphi(x)$ for $x \in V(Q) \setminus \{q_0\}$, $L_1(q_0) = (L(q_0) \setminus L(p_0)) \cup \{\varphi(q_1)\}$ and $L_1(x) = L(x)$ otherwise; G_1 cannot be L_1-colorable. Since a path $Q - q_0$ of length 4 is precolored in G_1 and $d(Q - q_0) \geq d(P) - 3 \geq r(P) - 3 = r(Q - q_0)$, the minimality of G implies that G_1 violates (Q) or (OBSTb). In the former case, as q_2 cannot be a vertex of degree two with a list of size three, (9) implies that G consists of H and a vertex with list of size two adjacent to q_2 and v_2, and it is L-colorable. Similarly, in the latter case, G_1 must be OBSTb2 and G is L-colorable. This is a contradiction. $\square$

Let H be one of the obstructions from Figure 1 or 2. A set $U \subseteq V(H)$ *has lists determined by the rest of H* if whenever L_1 and L_2 are two list assignments to H such that

- the size of the list of each vertex is given by Figure 1 or 2,
- $L_1(x) = L_2(x)$ for each $x \notin U$,
- vertices with list of size one give a proper coloring of the path induced by them, and
- H is neither L_1-colorable nor L_2-colorable,

then $L_1 = L_2$. That is, the list assignment that does not extend to H is uniquely determined once it is known on all the vertices except for those in U. We call H *k-determined* if every subset U of vertices of H of size at most k consisting only of vertices with list of size two has lists determined by the rest of H. A straightforward case analysis shows the following.

All graphs in Figures 1 and 2 are 1-determined. All except OBSTa2, OBSTx1c, OBSTx2b, OBSTb1, OBSTb1a, OBSTb3, OBSTb5 and OBSTb6 are 2-determined.

(20)

Let us now further discuss the subcase (A1) of (19); see Figure 6 for cases (B3) and (B4).

If $|X| = 3$ and $v_2v_3v_4z_2z_1$ is a 5-face, then there exists

(B1) a triangle incident with v_2, v_4, z_1 or z_2, or

(B2) a 4-face incident with z_1 or z_2, or

(B3) adjacent vertices $w_1, w_2 \in G - (X \cup \{z_1, z_2\})$ such that $w_1z_2, w_2v_5, w_2v_6 \in E(G)$, and furthermore, $|L(v_7)| = 2$, or

(B4) a path $w_1w_2w_3w_4w_5$ in $G - (X \cup \{z_1, z_2\})$ such that $w_2w_4 \in E(G)$, and either $v_2w_1, z_1w_3, z_2w_5 \in E(G)$ or $z_1w_1, z_2w_3, v_4w_5 \in E(G)$ (possibly with $w_1 = v_1$ in the former case or $w_5 = z_5$ in the latter case).

(21)

Proof. Suppose that none of these conditions is satisfied. Since v_2 and v_4 have list of size three, they must have degree at least three in G, and thus (18) implies that $z_1, z_2 \notin V(C)$, unless (B1) holds. Let φ be the coloring of X, $G' = G - X$ and L' the list assignment to G' as chosen in the proof of (19). Note that $|L'(z_1)|, |L'(z_2)| \geq 2$. As in the proof of (19), we conclude that $G' - \{z_1, z_2\}$ is L'-colorable.

There exist at least two L'-colorings φ_1 and φ_2 of the path z_1z_2 such that $\varphi_1(z_1) \neq \varphi_2(z_1)$ and $\varphi_1(z_2) \neq \varphi_2(z_2)$. For $i \in \{1, 2\}$, let L_i be the list assignment obtained from L' by removing the colors of z_1 and z_2 according to φ_i from the lists of their neighbors. Then (18) implies that L_i satisfies (S2), and by (10), (Q) holds as well.

Let G'' be the graph obtained from $G' - \{z_1, z_2\}$ by repeatedly removing the vertices whose degree is less than the size of their list both in L_1 and in L_2. Note that G'' is L_i-colorable if and only if G is L-colorable, for $i \in \{1, 2\}$. Let us argue that (I) is satisfied in G''. Unless (B1) or (B2) holds, (18) implies that no neighbor of z_1 and z_2 other than v_2 and v_4 lies in C, and furthermore, there exists no path wxy, where $w \in \{z_1, z_2\}$, $x \notin \{v_2, v_4, z_1, z_2\}$ and $|L(y)| = 2$. Thus, (I) holds unless there exists a path $wxyv$ with $w \in \{z_1, z_2\}$, $v \in \{v_2, v_4, z_1, z_2\}$ and $x, y \in V(G) \setminus (V(C) \cup \{z_1, z_2\}$. Since (B1) and (B2) are false, we have $w = z_1$ and $v = v_4$ or $w = z_2$ and $v = v_2$. However, then (5) implies that z_1 or z_2 has degree two, which is a contradiction.

Let us now consider the condition (T) for G''. Suppose that there exists a path $w_1w_2w_3w_4w_5$ with $w_2w_4 \in E(G)$ and $|L_i(w_1)| = |L_i(w_3)| = |L_i(w_5)| = 2$ for some $i \in \{1, 2\}$. If $|L(w_3)| = 2$, then by (T) and symmetry, we may assume that $|L(w_1)| = 3$, and thus $w_1 \notin \{v_1, v_5\}$ and by (18), $w_1 \notin V(C)$. Consider the (≤ 5)-chord Q contained in $X \cup \{z_1, z_2, w_1, w_2, w_3, w_4\}$ such that the subgraph F of G that is split off by Q contains neither P nor the triangle $w_2w_3w_4$. We have $d(Q) \geq B - 3 \geq r(Q)$ in F, since the triangle $w_2w_3w_4$ intersects Q. By the minimality of G and the choice of Q, we conclude that F violates (S3), (Q) or (OBSTb) (with the list assignment matching L on $V(F) \setminus V(Q)$ and an L-coloring of the rest of the graph on Q). If F violates (OBSTb), then by (5) and (9), F is isomorphic to one of the graphs in Figure 2. Since $|L(w_3)| = 2$, this is only possible if $\ell(Q) = 5$ and $w_5 \in V(F) \setminus V(Q)$. However, note that v_5 has degree two in F and thus it has degree one in $G - X$. It follows that $v_5 \notin V(G'')$, and similarly we conclude that $(V(F) \setminus V(Q)) \cap V(G'') = \emptyset$. This implies that $w_5 \notin V(G'')$, which is a contradiction. If F violates (S3) or (Q), then (5) and (9) imply that Q splits off a face. In particular, we have $v_4 \in V(Q)$. If (S3) fails, then we have that $v_5 = w_3$ and that w_1 is adjacent to z_2. Since w_1 has degree at least three, (5) implies that w_5 is not adjacent to v_2, z_1 or z_2; therefore, $|L(w_5)| = 2$, and by (9) we have $w_5 = v_7$ and G satisfies (B3). If (Q) fails, then note that v_5

has degree one in $G - X$, hence $v_5 \notin V(G'')$ and consequently, $v_5 \neq w_5$. It follows that v_5 is adjacent to w_2, and by (T), we have $|L(w_5)| = 3$. However, by symmetry of the path $w_1w_2w_3w_4w_5$, we conclude that v_5 is also adjacent to w_4, which is a contradiction since $v_5 \neq w_3$.

Suppose now that $|L(w_3)| = 3$ and w_3 has a neighbor in $X \cup \{z_1, z_2\}$. If $|L(w_i)| = 3$ or $w_i \in \{v_1, v_5\}$ holds for each $i \in \{1, 5\}$, then since both z_1 and z_2 have degree at least three, (5) implies that (B4) holds. Therefore, by symmetry we may assume that $|L(w_1)| = 2$ and $w_1 \notin \{v_1, v_5\}$. Again, we consider the (≤ 5)-chord Q contained in $X \cup \{z_1, z_2, w_1, w_2, w_3, w_4\}$ and the subgraph F of G that is split off by Q containing neither P nor the triangle $w_2w_3w_4$. As in the previous paragraph, we conclude that F is a face and violates (S3) or (Q). If $|L(w_5)| = 2$, then by symmetry we can assume that $w_5 \in V(F)$, and thus $w_5 = v_5$. However, in that case $w_5 \notin V(G'')$, which is a contradiction. Therefore, $|L(w_5)| = 3$ and $w_5 \notin V(F)$. Since z_1 has degree at least three, w_5 is adjacent to z_1 by (5). However, v_5 is adjacent to w_2, and the path $v_5w_2w_3w_4w_5$ satisfies (B4).

It follows that G'' satisfies (T). Let us now show that G'' is L_1-colorable or L_2-colorable, thus obtaining an L-coloring of G and a contradiction. Suppose first that neither z_1 nor z_2 have a neighbor in P. Then both L_1 and L_2 satisfy (S3). We conclude that G'' violates (OBSTa) or (OBSTb). Thus, G contains a (unique) near-obstruction H. The case that $|L_i(v)| = |L'(v)|$ for every $v \in V(G)$ is excluded similarly to (19), thus H has at least one vertex u_1 such that say $|L'(u_1)| = 3$ and $|L_i(u_1)| = 2$. Let K be the outer face of H, and let $q_0q_1 \ldots q_t = K - V(P)$, where q_0 is the neighbor of p_0 (or of p_1, if H is OBSTb1, OBSTb2 or OBSTx1 and $p_0 \notin V(H)$).

The vertex u_1 cannot be adjacent to both z_1 and z_2, thus $L_1(u_1) \neq L_2(u_1)$. Since H is neither L_1-colorable nor L_2-colorable and H is 1-determined by (20), it follows that H contains another vertex u_2 such that $|L'(u_2)| = 3$ and $|L_i(u_2)| = 2$. Suppose that u_1 and u_2 are both adjacent to z_1 or both adjacent to z_2. Since (B1) and (B2) are false, the distance between u_1 and u_2 must be at least three. Furthermore, we may assume that no other vertex between u_1 and u_2 in $K - V(P)$ has list of size two. This is only possible if H is OBSTa1, OBSTa5, OBSTx2a, or OBSTx3. Note that H is not OBSTa1, OBSTa5 or OBSTx3, since OBSTa1 is 2-determined and OBSTa5 and OBSTx3 are 4-determined. Therefore, either H is OBSTx2a or we may assume that u_1 is adjacent to z_1, u_2 is adjacent to z_2, and that $L_i(x) = L'(x)$ for $i \in \{1, 2\}$ and $x \in V(H) \setminus \{u_1, u_2\}$. In the latter case, we conclude that H is not 2-determined. By (20), H is one of OBSTa2, OBSTx1c, OBSTx2b, OBSTb1, OBSTb1a, OBSTb3, OBSTb5 or OBSTb6.

Let us make one more useful observation: suppose that $\ell(P) = 2$, q_0 is adjacent to p_0 and $|L_1(q_0)| = 2$. If $|L'(q_0)| = 3$, then consider the subgraph

G_2 of G that is split off by the path $Q = p_0q_0zv$, where $z \in \{z_1, z_2\}$ and $v \in \{v_2, v_4\}$. By the minimality of G, there exists an L-coloring of this path that does not extend to G_2. Since H contains a triangle whose distance to Q is at most 3, we conclude that G_2 violates (Q), and thus v_5 is adjacent to p_0. However, then $\ell(C) \leq 8$, contradicting (8). Therefore, $|L'(q_0)| = 2$, and by (18), if (B1) and (B2) are false, then $|L(q_0)| = 2$. Since u_1 and u_2 exist, in this situation H has at least three vertices with list of size two. This implies that H is neither OBSTa2 not OBSTx1c. It also implies that H is not OBSTx2a, since OBSTx2a is 2-determined.

Let us consider other obstructions separately:

- *H is OBSTx2b:* If p_0 has degree two in H, then by the observation we have $|L(q_0)| = 2$, and thus H is a tame near-obstruction, contradicting (12). Thus, p_0 has degree three in H. Furthermore, (12) implies that $q_5 \notin V(C)$, and thus q_5 is adjacent to z_1 and q_3 is adjacent to z_2. If $|L(q_1)| = 2$, then by (11) applied to (a subpath of) $v_4z_2q_3q_2q_1$, v_5 is adjacent to q_2 (possibly $v_5 = z_1$). However, by (5) and (9) G does not contain any other vertices, and such a graph is L-colorable. Thus, $|L(q_1)| = 3$ and q_1 is adjacent to v_4. By (11) for $p_0q_0q_1v_4$, we conclude that v_5 is adjacent to p_0, contradicting (8).

- *H is OBSTb1 or OBSTb1a:* If $p_0 \in V(H)$, then by (11) for the path $v_4z_2u_2p_0$, we have that v_5 is adjacent to p_0. However, then G contains no other vertices and is L-colorable. Thus, $p_0 \notin V(H)$ and H is OBSTb1. In this case, we similarly conclude that the path $p_0p_1u_2z_2v_4$ splits off a face, OBSTb1 or OBSTb2. In all these cases, G is L-colorable.

- *H is OBSTb3:* This is excluded by (10).

- *H is OBSTb5:* Suppose that $u_2 = q_0$. Then $u_1 = q_2$ and $q_4 = v_1$, and by (11) applied to $v_4z_2q_0p_0$, we conclude that v_5 is adjacent to p_0. However, such a graph is L-colorable. So, $u_2 = q_2$ and $u_1 = q_4$. If $|L(q_0)| = 3$, then q_0 would be adjacent to v_4, contradicting (18). Thus, $|L(q_0)| = 2$. Consider the path $q_0q_1q_2z_2v_4$. By (11), v_5 is adjacent to q_1 (possibly $v_5 = q_0$). However, then G is L-colorable.

- *H is OBSTb6:* Let us note that only one two-element subset of vertices of H with list of size two does not have lists determined by the rest of H—the one consisting of the two rightmost square vertices in the depiction of OBSTb6 in Figure 2). So, we may assume that p_3 has degree 4 in H, $u_2 = q_4$ and $u_1 = q_6$, and $|L(q_0)| = 2$. If v_4 is adjacent to q_2, then considering the subgraph split off by the path $q_0q_1q_2v_4$, we conclude that $v_5 = q_2$ and $|L(q_2)| = 2$. If v_4 is not adjacent to q_2, then $|L(q_2)| = 2$ as well. By (11) applied to $q_2q_3q_4z_2v_4$, we have that v_5 is adjacent to q_3. And again, we conclude that G is L-colorable.

Let us now consider the case that z_1 or z_2 is adjacent to a vertex of P. By (18), this vertex must be an internal vertex of P. If exactly one of z_1 and z_2 has a neighbor in P, then by (10) at least one of L_1 and L_2, say L_1, satisfies (S3). It follows that G'' with the list assignment L_1 must violate (OBSTa) or (OBSTb), and contains a near-obstruction H. However, since one of z_1 and z_2 has an internal vertex $p \in P$ as a neighbor, p is a cut-vertex in G'', thus this is only possible if $p \in \{p_1, p_{m-1}\}$ and either $\ell(P) = 2$ and H is OBSTx1, or $\ell(P) = 5$ and H is OBSTb1 or OBSTb2. Suppose that there exists a vertex $v \in V(H)$ adjacent to p such that $|L_1(v)| = 2$. By (7), v is adjacent to v_2, v_4, z_1 or z_2. Since z_1 or z_2 is adjacent to p and neither z_1 nor z_2 is incident with a (≤ 4)-cycle, (5) implies that z_1 or z_2 has degree two. This is a contradiction. It follows that no vertex with list of size two is adjacent to p, hence $\ell(P) = 2$. By (12), the two vertices of H with list of size two are adjacent to z_2 and v_4, respectively. However, then p_0 and v_4 are joined by a 2-chord contradicting (18).

Finally, suppose that both z_1 and z_2 have a neighbor in P. Since neither (B1) nor (B2) holds, the neighbors of z_1 and z_2 are internal vertices of P by (18), and $\ell(P) \geq 4$. Let p_i be the neighbor of z_1 and p_j the neighbor of z_2. Suppose that $i < m-1$ or $j < m-3$. By (5), P contains two adjacent vertices of degree two that are not contained in any (≤ 5)-cycle. In that case, contract these two vertices into one (and change its color so that it is consistent with the colors of its neighbors). The resulting graph is a smaller counterexample to Theorem 2, which is a contradiction. Therefore, $i = m-1$ and $j = m-3$. Let $Q = p_0p_1 \ldots p_{m-3}z_2v_4$, and let φ be an L-coloring of the subgraph of G induced by $V(P) \cup X \cup \{v_1, z_1, z_2\}$ that exists by the minimality of G. Let $G_3 = G - (V(P) \setminus V(Q)) - \{v_1, v_2, v_3, z_1\}$. Let L_3 be the list coloring such that $L_3(x) = \varphi(x)$ for $x \in V(Q)$ and $L_3(x) = L(x)$ otherwise. The graph G_3 is not L_3-colorable, thus it violates (Q) or contains OBSTb1 or OBSTb2. If G_3 violates (Q), then (18) implies that v_5 is adjacent to p_0 and G contains OBSTb2 or OBSTb2a. If G_3 contains OBSTb1, then G contains OBSTb6. Otherwise, G is L-colorable. $\square$

Let T be the 4-cycle in distance at most one or a triangle in distance at most two from X, which exists by (19) and (21). Since $d(P, T) \leq 4$, we have $\ell(P) = 2$.

Suppose that (A3) happens, i.e., T is a 4-cycle sharing an edge with the path $p_2v_1 \ldots v_k$. Let v_iv_{i+1} be such an edge with i minimal and let φ be an L-coloring of the path $p_2v_1 \ldots v_i$. Let G' be the graph obtained from

$G - v_i v_{i+1}$ by adding a vertex v adjacent to v_i and v_{i+1}. Let c be a color that does not appear in the lists of v_i and v_{i+1}. Let L' be a list assignment such that $L'(x) = \{\varphi(x)\}$ for $x \in \{v_1, \ldots, v_i\}$, $L'(v) = \{c\}$ if $|L(v_{i+1})| = 2$ and $L'(v) = \{\varphi(v_i), c\}$ if $|L(v_{i+1})| = 3$, $L'(v_{i+1}) = (L(v_{i+1}) \setminus \{\varphi(v_i)\}) \cup \{c\}$ and $L'(x) = L(x)$ for other vertices $x \in V(G')$. Note that G' is not L'-colorable. Furthermore, by the choice of X, if $k = 4$ then $|L(v_k)| = 4$, hence a path R of length at most 5 is precolored in P. Furthermore, since T contains the edge $v_i v_{i+1}$, we have $d(R) \geq B - 5 \geq r(R)$. By (7) and (18), R is an induced path and no vertex with list of size two other than v_s, v_{i+1} and v is adjacent to it, and since $\ell(C) \geq 9$, it follows that (S3) and (Q) are satisfied. Since T is a 4-cycle, v cannot be in distance at most one from a triangle in G', thus (T) holds as well. By the minimality of G, we conclude that G' violates (OBSTb); let H be the minimal non-L'-colorable subgraph of G'. We have $\ell(R) \geq 4$, and consequently, $i \geq 1$. If $i = 1$, then we also have $|L'(v)| = 1$, $|L(v_2)| = 2$ and $|L(v_1)| = 3$; let $w = v_1$. If $i \geq 2$, then choose $w \in \{v_1, v_2\}$ such that $|L(w)| = 3$. Such a vertex w has degree at least three in G, and thus it has degree at least three in H (even if w is an endvertex of the precolored path of H, since then w has a neighbor x with list of size two in H, and the edge wx belongs to C by (18)). There exist L-colorings φ_1 and φ_2 of the path $p_2 v_1 \ldots v_i$ such that $\varphi_1(w) \neq \varphi_2(w)$; let L'_1 and L'_2 be the corresponding list assignments to G'. Since G' is neither L'_1-colorable nor L'_2-colorable, the inspection of the graphs in Figure 2 shows that H is OBSTb1, OBSTb1a, OBSTb1b, OBSTb3 or OBSTb5. Since the edge $v_{i-1}v_i$ is not incident with T, the vertex v_i has degree at least three in G, and hence also in H; therefore, H is OBSTb3 and $|L'(v)| = 1$. However, (5) and (9) imply that $V(G) = V(H) \setminus \{v\}$, contradicting (8). We conclude that (A3) is false.

Now, suppose that (B2) happens. If $v_4 \in V(T)$, then let $Y = \{v_3, v_4\}$. If $v_4 \notin V(T)$ and $z_2 \in V(T)$, then let $Y = \{v_3, v_4, z_2\}$; otherwise let $Y = \{v_3, v_4, z_2, z_1\}$. Note that if $z_1 \in Y$, then z_2 is not incident with a 4-cycle, and since (A3) is false, at most one of z_1 and z_2 has a neighbor in P. Thus, there exists an L-coloring ψ of the subgraph G_0 of G induced by $Y \cup V(P) \cup \{z_1, v_1, v_2\}$ such that $\psi(v_4) \notin L(v_5)$. Let $G' = G - Y$ and let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if $x \in V(G') \setminus \{v_1, v_2\}$ has a neighbor $y \in Y$, and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. Since z_2 has degree at least three, (5) and (18) together with the choice of Y imply that G' satisfies (I) and (S2). Obviously, (T) is satisfied as well. Suppose that a vertex v with $|L'(v)| = 2$ has two neighbors in $p_0 p_1 p_2 v_1 v_2$. By (7), we have $|L(v)| = 3$, hence v is adjacent to a vertex in Y. Suppose that $v \neq z_1$. Since (A3) is false, v is not adjacent to z_1; but then (5) implies that z_1 has degree two, which is a contradiction. Therefore, $v = z_1$, and since ψ assigns a color to z_1, G' satisfies (Q). Hence, G' violates (OBSTb); let H be the subgraph of G' isomorphic to OBSTb1 or OBSTb2. Note that v_2 is adjacent to a vertex x such that $|L'(x)| = 2$. Since z_1 has degree at least three, (5) implies that $x = z_1$, and thus $Y = \{v_3, v_4, z_2\}$. Furthermore, note that neither z_1 nor z_2 has a neighbor in P, thus there exists an L-coloring ψ' of the subgraph of G_0 such that $\psi'(y) = \psi(y)$ for $y \in \{v_1, v_2, v_3, v_4\}$ and $\psi'(z_2) \neq \psi(z_2)$. Since both OBSTb1 and OBSTb2 are 1-determined, z_2 has a neighbor in H different from z_1. Furthermore, H is not OBSTb2, since OBSTb2 is 2-determined and z_2 cannot have more than two neighbors in H whose list according to L' has size two. However, if H is OBSTb1, then p_0 and v_4 are joined by a 3-chord, and by (11), v_5 is a common neighbor of p_0 and v_4. This contradicts (8).

Therefore, neither (A3) nor (B2) holds and T is a triangle. Let us consider the case that (B4) is true.

- *Suppose first that $v_2w_1, z_1w_3, z_2w_5 \in E(G)$.* Note that v_1 may be equal to w_1. Let $S = L(v_2) \setminus (L(v_1) \setminus L(p_2))$. If $S \not\subseteq L(z_1)$, then let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_2)$, $L'(v_2) = S \setminus L(z_1)$ and $L'(x) = L(x)$ otherwise. Observe that the graph $G - \{z_1, w_3\}$ is not L'-colorable and that it satisfies the assumptions of Theorem 2 (it satisfies (OBSTb), since v_3 is the only neighbor of v_2 with list of size two and $v_1v_2v_3$ cannot be a subpath of a 5-cycle), contradicting the minimality of G. Thus, $S \subseteq L(z_1)$. If $S \neq L(v_3)$, then choose a color $c \in S \setminus L(v_3)$; let L' be the list assignment obtained from L by removing c from the lists of neighbors of v_2 other than v_1. Note that $G - v_2$ is not L'-colorable, and as in (19), we conclude that $G - v_2$ is a smaller counterexample to Theorem 2, which is a contradiction. Similarly, we exclude the case that a color $c' \in L(v_4) \setminus L(v_5)$ does not belong either to S or to $L(z_2)$. Therefore, there exists a color $c' \in S \cap L(z_2)$. By (5) and (18), z_2 is not adjacent to a vertex of P.

  Suppose that w_1 and w_5 do not have a common neighbor. Let G' be the graph obtained from $G - \{w_3, z_1, v_3\}$ by identifying v_2 with z_2 to a new vertex v. Let L' be the list assignment such that $L'(v_1) = L(v_1) \setminus L(p_2)$, $L'(v) = \{c'\}$, $L'(v_4) = \{c''\}$ for a color $c'' \in L(v_4) \setminus \{c'\}$ such that $L(v_3) \neq \{c', c''\}$ and $L'(x) = L(x)$ otherwise. Note that $t(G') \geq B$, since both v_2 and z_2 are in distance at least $B - 2$ in G from any (≤ 4)-cycle different from T. Since w_1 and w_5 do not have a common neighbor, (5) implies that v is not contained in any (≤ 4)-cycle in G'. Since G' is not L'-colorable, we conclude that it violates (OBSTb). Let H be the subgraph of G' isomorphic to one of the graphs in Figure 2.

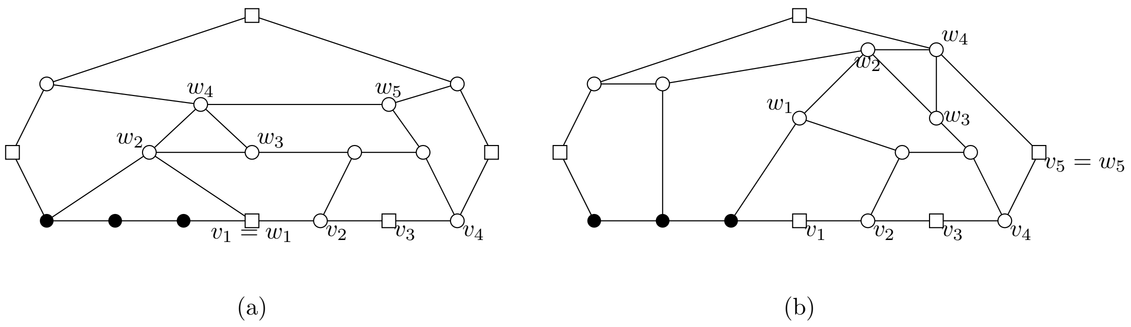


Figure 7: Configuration in case that (B4) holds.

By (7), p_1 is not adjacent to a vertex with list of size two, hence v_4 belongs to H. Note that v has degree at least three in H, as otherwise G contains a cycle K of length at most 7 such that $v_1v_2v_3v_4 \subset K$ and the open disk bounded by K contains z_1, z_2 and w_3, contradicting (5). The inspection of the graphs in Figure 2 shows that v has degree exactly three and that both internal faces incident with v in H have length five. Similarly, (5) implies that $vw_5 \in E(H)$ and $w_1 = v_1$. But then $v_1vw_5w_4w_2$ is the only 5-cycle in G' containing the edge v_1v, thus $v_1w_2 \in E(H)$ and v_1 has degree at least three in H. This is only possible if H is OBSTb4. However, then H is the graph in Figure 7(a), which is L-colorable.

So, suppose that w_1 and w_5 have a common neighbor w, and thus by (5), w_2 and w_4 have degree three. By (18), $|L(w)| = 3$. Let ψ be an L-coloring of $p_2v_1v_2v_3v_4z_2$ such that $\psi(v_4) = c'$. Let d be a color in $L(z_1) \setminus \{\psi(v_2), \psi(z_2)\}$. Note that z_2 has no neighbor in P by (5). If $w_1 \neq v_1$, then let d' be a color in $L(w_1) \setminus \{\psi(v_2)\}$ such that $L(w_2) \setminus \{d'\} = L(w_3) \setminus \{d\}$, if such a color exists, and an arbitrary color in $L(w_1)$ otherwise. Among the possible choices of ψ, d and d', we choose them so that the following additional conditions hold:

- If w_1 is adjacent to p_1, then $L(w_1) \neq L(p_2) \cup \{\psi(v_2), d'\}$.
- If $w_1 = v_1$, then either $\psi(v_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi(v_1)\} \neq L(w_3) \setminus \{d\}$.
- If $w_1 \neq v_1$, w_1 is not adjacent to p_1 and p_1 has a neighbor $z \notin V(C)$, then $L(z) \setminus L(p_1) \neq L(w_5) \setminus \{\psi(z_2)\}$.

Let $G' = G - \{w_2, w_3, w_4, z_1, z_2, v_3, v_4\}$, with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(w_1) = L(w_1) \setminus \{d'\}$ if $w_1 \neq v_1$, $L'(x) = L(x) \setminus \{\psi(y)\}$ for every vertex x with a neighbor $y \in \{v_4, z_2\}$ and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable. If w_1

had a common neighbor with v_4 or z_2, then (5) would imply that w has degree two; hence (18) implies that G' satisfies (I). If G' violated (Q), then (5) and (18) would imply that w_1 is adjacent to p_1. But, in that case the choice of ψ, d and d' ensures that (Q) holds. Hence, G' violates (OBSTb) and contains a subgraph H isomorphic to OBSTb1 or OBSTb2. Then v_2 is adjacent to a vertex with list of size two, and by (5), this vertex is w_1; hence, we have $w_1 \neq v_1$. Note that there exists a path w_1xy in H such that y has list of size two. By (18), we have $|L(y)| = 3$, hence y is adjacent to z_2 or v_4. Since w has degree at least three, (5) implies $x = w$ and $y = w_5$. If H were OBSTb1, then w_5 would be adjacent to p_0, and by (11) applied to $v_4z_2w_5p_0$, we would have that v_5 is adjacent to p_0, contradicting (8). It follows that H is OBSTb2. Note that w_1 is not adjacent to p_1, thus the unique neighbor z of p_1 in $V(H) \setminus V(C)$ satisfies $L'(z) \setminus L(p_1) \neq L'(w_5)$. However, then H is L'-colorable, contradicting the assumption that (OBSTb) does not hold.

- *Next, consider the case that* $z_1w_1, z_2w_3, v_4w_5 \in E(G)$. Note that w_5 may be equal to v_5. Similarly to the previous case, we conclude that $L(v_2)\setminus(L(v_1)\setminus L(p_2)) = L(v_3) \subseteq L(v_4)$, that each color $c' \in L(v_4)\setminus L(v_5)$ belongs to both $L(v_3)$ and $L(z_2)$ and that $L(z_1) = L(z_2)$—otherwise, we can color a subset Y of $X \cup \{z_2\}$, remove the colors of the vertices of Y from the lists of their neighbors, and obtain a smaller counterexample to Theorem 2.

  If $L(z_2) \neq L(v_4)$, then let ψ be an L-coloring of $p_2v_1v_2v_3v_4$ such that $\psi(v_4) \notin L(z_2)$. Let G' be the graph obtained from $G - \{v_3, z_2, w_3\}$ by adding the edge v_2v_4. Let c be a color that does not appear in any of the lists and L' the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2\}$, $L'(v_4) = \{c\}$, $L'(x) = (L(x) \setminus \{\psi(v_4)\}) \cup \{c\}$ for all other vertices x adjacent to v_4, and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. Also, by (5), the edge v_2v_4 is not incident with a (≤ 4)-cycle, and thus $t(G') \geq B$. Furthermore, the distance from v_2 to T in G is three, thus $r(p_0p_1p_2v_1v_2v_4) \geq B - 7 \geq r(5)$. Since v_2 is not incident with a vertex with list of size two and every cycle containing the edge v_2v_4 has length at least seven, G' satisfies (OBSTb) and contradicts the minimality of G.

  Therefore, $L(z_2) = L(v_4)$. If p_1 is adjacent to z_1, then let $G' = G - \{p_2, v_1, v_2, v_3, v_4, z_2, w_3\}$. Let ψ be an L-coloring of the subgraph of G induced by $\{p_1, p_2, v_1, v_2, v_3, v_4, z_1, z_2, w_1, w_2\}$ such that $\psi(v_4) \notin L(v_5)$ and $\psi(w_2) \notin L(w_3) \setminus \{\psi(z_2)\}$. Let L' be the list assignment such that

$L'(x) = \{\psi(x)\}$ for $x \in \{z_1, w_1, w_2\}$, $L'(x) = L(x) \setminus \{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. By (5), neither w_1 nor w_2 has a common neighbor with v_4 (since if $w_5 \neq v_5$, then w_5 has degree at least three). By (18), w_1 has no neighbor with list of size two in G', and since w_1 has degree at least three, (5) implies that G' satisfies (Q). Since G' is not L-colorable, by the minimality of G we conclude that G' violates (OBSTb). Because w_1 has degree at least three, (5) implies that G' contains OBSTb2. Let y be the neighbor of w_2 with list of size two and consider the path $Q = v_4w_5w_4w_2y$. If Q is not a subpath of C, then v_4 and w_2 have a common neighbor by (11), implying that w_2 has degree two, which is a contradiction. Therefore, $w_5 = v_5$ and $Q \subset C$. However, then there exists an L-coloring ψ' of the subgraph of G split off by the 3-chord $p_1z_1w_1w_2$ that differs from ψ exactly in the colors of w_1 and w_2, and at least one of ψ and ψ' extends to an L-coloring of G. This is a contradiction.

It follows that $p_1z_1 \notin E(G)$. Suppose now that w_1 and w_5 do not have a common neighbor. Then, let G' be the graph obtained from $G - \{v_3, z_2, w_3\}$ by identifying z_1 with v_4 to a new vertex v, with the list assignment L' such that $L'(v) = L(v_4) \setminus L(v_3)$, $L'(v_1) = L(v_1) \setminus L(p_2)$, $L'(v_2) \subseteq L(v_2) \setminus L'(v_1)$ has size one and $L'(x) = L(x)$ otherwise. Observe that G' satisfies $t(G') \geq B$ and that it is not L'-colorable. Also, since p_1 is not adjacent to z_1, (18) implies that G' satisfies (S3). No vertex with list of size two is adjacent to p_1 or v_2 and the only vertex with list of size two adjacent to v is v_5, thus G' satisfies (Q). We conclude that G' violates (OBSTb); let H be the subgraph of G' isomorphic to one of the graphs drawn in Figure 2. By (18), v_2 has degree two in H. If v had degree two, then $v_1v_2v_3v_4$ would be a subpath of a cycle K of length at most seven in G, such that the open disk bounded by K contains z_1, z_2 and w_3. This contradicts (5), hence v has degree three in H and H is OBSTb4. Let x be the common neighbor of p_2 and v in H. By (18), x is adjacent to z_1 in G. In H, there exists a path $xyzv_5$, and by (5) we have $x = w_1$, $y = w_2$, $z = w_4$ and $v_5 = w_5$. Then G is the graph depicted in Figure 7(b), which is L-colorable.

Therefore, w_1 and w_5 have a common neighbor w. By (18), $|L(w)| = 3$, and by (5), w_2 and w_4 have degree three. Suppose now that w_1 has no neighbor in P. Then there exists an L-coloring ψ of the subgraph G_0 of G induced by $V(P) \cup \{v_1, v_2, v_3, v_4, z_1, z_2, w_1\}$ such that $\psi(v_4) \notin L(v_5)$ and either $\psi(w_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi(w_1)\} \neq L(w_3) \setminus \{\psi(z_2)\}$. Let $G' = G - \{v_3, v_4, z_2, w_2, w_3, w_4\}$ with the list assignment L' such that

$L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, z_1\}$, $L'(w_1) = \{\psi(z_1), \psi(w_1)\}$, $L'(x) = L(x) \setminus \{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable. By (5) and (18), G' satisfies (I), and since p_1 is not adjacent to z_1, G' satisfies (S3). Since w_1 has no neighbor in P and v_2 has no neighbor with list of size two, G' also satisfies (Q). We conclude that (OBSTb) is violated and that G' contains one of the graphs depicted in Figure 2; let H be such a subgraph. The inspection of such graphs shows that if v_2 has degree three in H, then it is incident with a path v_2xyz with $|L'(z)| = 2$, where $z \neq w_1$. By (5), z is not a neighbor of v_4, hence $|L(z)| = 2$. However, that contradicts (18). Therefore, v_2 has degree two in H. Similarly, we conclude that v_1 has degree two in H, thus H is OBSTb1a, OBSTb1b or OBSTb4. Note that there exists an L-coloring ψ' of G_0 such that ψ' matches ψ on v_1, v_2, v_3 and v_4, either $\psi'(w_1) \notin L(w_2)$ or $L(w_2) \setminus \{\psi'(w_1)\} \neq L(w_3) \setminus \{\psi'(z_2)\}$, and $\psi'(z_1) \neq \psi(z_2)$ (ψ' may or may not differ from ψ on w_1). Note that ψ' does not extend to a coloring of H; that is only possible if H is OBSTb1a and $\psi(w_1) = \psi'(w_1)$. But then there exists a path $v_2z_1xyp_0$ with $|L'(y)| = 2$. By (18), we have $|L(y)| = 3$, thus y is adjacent to v_4. However, then v_4yp_0 is a 2-chord contradicting (18).

Finally, consider the case that w_1 has a neighbor $p_i \in V(P)$. By (5), z_1 has degree three. Observe that there exist colors $c_1 \in L(w_1) \setminus L(p_i)$ and $c_2 \in L(v_2) \setminus (L(v_1) \setminus L(p_2))$ such that $c_1 = c_2$ or $c_1 \notin L(z_1)$ or $c_2 \notin L(z_1)$. Let G' be the graph obtained from $G - \{p_{i+1}, \ldots, p_2, v_1, z_1, z_2, w_2, w_3, w_4\}$ by identifying w_1 with v_2 to a new vertex v. By (5), v is not incident with a (≤ 4)-cycle, thus $t(G') \geq B$ and $d(p_0 \ldots p_i v) \geq B - 4 > r(3)$. Let c be a new color that does not appear in any of the lists and L' the list assignment such that $L'(v) = \{c\}$, $L'(v_3) = (L(v_3) \setminus \{c_2\}) \cup \{c\}$, $L'(x) = (L(x) \setminus \{c_1\}) \cup \{c\}$ if x is a neighbor of w_1 and $L'(x) = L(x)$ otherwise. Observe that G' is a counterexample to Theorem 2 smaller than G, which is a contradiction.

Therefore, (B4) is false.

Suppose that (A4) holds. Note that $w_1 \neq v_1$ and $w_5 \neq v_5$, since v_2 and v_4 have list of size three. Suppose first that there exists an L-coloring ψ of the subgraph induced by $V(P) \cup \{v_1, v_2, v_3, v_4, w_1, w_2\}$ such that $\psi(v_4) \notin L(v_5)$ and $|L(w_3) \setminus \{\psi(v_3), \psi(w_2)\}| \geq 2$. Then, let $G' = G - \{v_3, v_4, w_3\}$ with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, w_1\}$, $L'(w_2) = \{\psi(w_1), \psi(w_2)\}$, $L'(x) = L(x) \setminus \{\psi(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. Note that G' is not L-colorable, and the choice of ψ ensures that (S3) holds. By (5), no neighbor of w_2 is adjacent to v_4, as otherwise w_5 would have degree two; thus, (18) implies that (I) holds. As w_1

has degree at least three, (5) implies that w_2 is not adjacent to a vertex of P and (Q) holds. Therefore, G' violates (OBSTb) and contains a subgraph H isomorphic to one of the graphs drawn in Figure 2. No neighbor of v_2 has list of size two, thus w_1 belongs to H. If v_1 or v_2 had degree greater than two in H, then G would contain a (≤ 3)-chord contradicting (9) or (18); hence, H is OBSTb1a, OBSTb1b or OBSTb4. Since w_1 has degree at least three, H is not OBSTb1a. If H were OBSTb1b, then G would contain a (≤ 3)-chord starting in v_2 contradicting (18). Finally, if H is OBSTb4, then let w_2yz be the path in the boundary of the outer face of H with $|L'(z)| = 2$. If z is a neighbor of v_4, then by (5) we have $y = w_4$ and $z = w_5$; however, then there exists a path $v_4w_5y'z'$ in the boundary of the outer face of H with $|L(z')| = 2$, contradicting (18). Otherwise, we have $|L(z)| = 2$. Consider the subgraph split off by $v_3w_3w_4w_2yz$. Since both v_3 and z have list of size two and w_3 and y have no common neighbor, this subgraph satisfies the assumptions of Theorem 2, contradicting the minimality of G.

Suppose now that such a coloring ψ does not exist. (5) and (18) show that can only happen if w_1 is adjacent to p_1. Since w_5 has degree at least three, (11) implies that w_4 has no neighbor in P. Let ψ' be an L-coloring of the subgraph induced by $V(P) \cup \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4\}$ such that $\psi'(v_4) \notin L(v_5)$, $G' = G - \{p_2, v_1, v_2, v_3, w_3\}$, $L'(x) = \{\psi'(x)\}$ for $x \in \{w_1, w_2, w_4\}$, $L'(x) = L(x) \setminus \{\psi'(v_4)\}$ if x is a neighbor of v_4 and $L'(x) = L(x)$ otherwise. By (5) and (18), w_2 is not adjacent to any vertex with list of size two and w_5 is the only neighbor of w_4 with list of size two. Furthermore, w_5 is not adjacent to p_0 by (18), and it is not adjacent to p_1, since (similarly to (11)) we would have that the path $p_0p_1w_5v_4$ splits off a 5-face, implying that v_5 is adjacent to p_0 and contradicting (8). It follows that G' satisfies (Q). Furthermore, G' satisfies (OBSTb), since by (18) it does not contain a path v_4w_5xy with $|L(y)| = 2$. Therefore, G' a counterexample to Theorem 2 smaller than G, which is a contradiction. Therefore, (A4) is false.

Suppose now that (B3) holds. Let ψ be an L-coloring of the subgraph G_0 of G induced by $V(P) \cup \{v_1, v_2, \ldots, v_6, w_2\}$ such that $\psi(v_6) \notin L(v_7)$ (w_2 has no neighbor in P by (5), thus such a coloring exists). Let L' be the list assignment such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, v_3\}$, $L'(v_4) = \{\psi(v_3), \psi(v_4)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in \{w_2, v_6\}$ and $L'(x) = L(x)$ otherwise. The graph $G' = G - \{v_5, v_6, w_2\}$ is not L'-colorable, and by (5) and (9), it satisfies (I) and (Q). Furthermore, note that there exists another L-coloring ψ' of G_0 such that $\psi'(v_6) = \psi(v_6)$, $\psi'(w_2) = \psi(w_2)$, $\psi'(v_4) \neq \psi(v_4)$ and $\psi'(v_2) \neq \psi(v_2)$, thus we can choose ψ so that (OBSTb) holds, unless G' contains OBSTb3. By (5) and (18), we then have that z_1 is adjacent to p_1 and w_1 is adjacent to p_0, and by (11) applied to $v_6w_2w_1p_0$, v_7 is adjacent to p_0. Nevertheless, such a graph is L-colorable. Therefore, G' contradicts the minimality of G. It follows that (B3) is false as well, hence

G satisfies (A2), (A5) or (B1).

(22)

Suppose that there exists a vertex $t \in V(T) \cap (V(P) \cup \{v_1\})$. Let G' be the graph obtained from G by splitting t to two vertices t' and t'' and adding a new vertex v adjacent to t' and t'', so that T becomes a 5-face. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup \{t\}$, c a color that does not appear in any of the lists, and let L' be the list assignment such that $L'(t') = L'(t'') = \{\psi(t)\}$, $L(v) = \{c\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, thus it must violate (OBSTb); let H be the subgraph of G' isomorphic to one of the graphs in Figure 2. In H, v has degree two and is incident with a 5-face. If $t \in V(P)$, then H is OBSTb1 or OBSTb2; but then G contains OBSTx1c or OBSTa6. Therefore, $t = v_1$. If H is OBSTb1, then G contains OBSTx1; if H is OBSTb1a, then G contains OBSTx1a; if H is OBSTb1b, then G contains OBSTx1b; if H is OBSTb2b, then G contains OBSTx4; and if H is OBSTb5, then G contains OBSTx2b. It follows that H is OBSTb4 or OBSTb6. By (5) and (9), we conclude that G is equal to the graph obtained from H by removing v and identifying t' with t''. However, then G is L-colorable. Therefore,

$V(T) \cap (V(P) \cup \{v_1\}) = \emptyset$.

(23)

Let X' be the subset of $\{v_s, v_{s-1}, v_{s-2}, v_{s-3}\}$ defined symmetrically to X on the other side of P. By symmetry and the assumption that $t(G) \geq B$, we conclude that T is also incident with a vertex of X' (the case (A2)) or one of the vertices z'_1 or z'_2 incident with the 5-face $v_{s-1}v_{s-2}v_{s-3}z'_2z'_1$ (the cases (B1) and (A5)). Let b be the first vertex in the sequence v_2, v_3, z_1, z_2 and v_4 that is incident with T, and let b' be the first such vertex among v_{s-1}, v_{s-2}, z'_1, z'_2 and v_{s-3}. Note that either $b = b'$ or b and b' are adjacent.

Suppose now that $V(T) \subseteq V(C)$. In this case (A5) does not hold. By (15), we have $b \in \{v_3, v_4\}$ and $b' \in \{v_{s-2}, v_{s-3}\}$. If $b' = v_{s-3}$, then $v_{s-3} \in X'$ and by the choice of X', we have $|L(v_{s-2})| = 2$. This contradicts (15). Thus $b' = v_{s-2}$ and symmetrically, $b = v_3$. By (15), we have $|L(v_2)| = |L(v_{s-1})| = 3$, and by (16), $|L(v_1)| = 2$. However, then $X = \{v_1\}$ and $b \notin X$, which is a contradiction. It follows that

T shares at most two vertices with C.

(24)

Suppose that $v_{s-3} \in X' \cap V(T)$ and $v_{s-2} \notin V(T)$. The choice of X' implies that $|L(v_{s-3})| = 3$ and $|L(v_{s-2})| = |L(v_{s-4})| = 2$. If $\{v_2, v_3, v_4\} \cap V(T) = \emptyset$, then $b \in \{z_1, z_2\}$; let $v \in \{v_2, v_4\}$ be the neighbor of b. By (18) applied to vbv_{s-3}, we conclude that $T = vbv_{s-3}$, contrary to the assumption that $v \notin V(T)$. It follows that a vertex of $\{v_2, v_3, v_4\} \cap V(T)$ is equal to either v_{s-3} or v_{s-4}. By (8), we have $6 \le s \le 8$. If $s = 8$, then $v_4 = v_{s-4}$, which is only possible if both X and X' satisfy (A5). Let $z_1z_2z_3$ be the path such that $T = z_2v_4v_5$, z_1 is adjacent to v_2 and z_3 is adjacent to v_7. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup \{v_1, v_2, v_3, v_6, v_7, v_8\}$ such that $\psi(v_3) \notin L(v_4)$ or $\psi(v_6) \notin L(v_5)$ or $L(v_4) \setminus \{\psi(v_3)\} \neq L(v_5) \setminus \{\psi(v_6)\}$. Let $G' = G - \{v_3, v_4, v_5, v_6\}$ with the list assignment L' such that $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_8\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(v_7) = \{\psi(v_7), \psi(v_8)\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and since v_2 and v_7 are the only vertices with list of size two, it is easy to see that it satisfies the assumptions of Theorem 2. This contradicts the minimality of G.

Therefore, $s \le 7$. By (7) and (24), C has no chords. If $t \in V(T) \setminus V(C)$ has a neighbor $v \in V(C)$, then vt is an edge of T, as otherwise (5) would imply that v_{s-1} or v_{s-5} (which have lists of size three) has degree two. Note that there exists at most one vertex with two neighbors in the path $p_0p_1p_2v_1v_2$ and another neighbor in T. If such a vertex v exists, then v_{s-4} has degree two by (5), hence $V(T) \cap V(C) = \{v_{s-3}\}$. Therefore, there exists an L-coloring ψ of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, \ldots, v_{s-4}, v\}$ such that $\psi(v_{s-3}) \notin L(v_{s-2})$. Let $G' = G - V(T)$ and let L' be the list assignment given by $L'(x) = \{\psi(x)\}$ for $x \in \{v_1, v_2, \ldots, v_{s-5}\}$, $L'(v_{s-4}) = \{\psi(v_{s-5}), \psi(v_{s-4})\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, and by (5) and (18), it satisfies (I). The choice of ψ ensures that (Q) holds as well. Thus, G' must violate (OBSTb), and in particular $s = 7$ and $v_3 \notin V(T)$. Let H be the subgraph of G' isomorphic to OBSTb1 or OBSTb2. By (5), v_s is the only vertex with list of size two adjacent to p_0, thus $v_s \in V(H)$. Let v_sxy be the path in the outer face of H such that $|L'(y)| = 2$. By (5), we have $x = v_{s-1}$ and $y = v_{s-2}$. hence H is OBSTb2. But then there exists a path of length three joining v_2 with v_{s-2} and contradicting (18). Therefore, if $v_{s-3} \in X' \cap V(T)$, then $v_{s-2} \in V(T)$, and in particular, $b' \neq v_{s-3}$. Symmetrically,

if $v_4 \in X \cap V(T)$, *then* $v_3 \in V(T)$, (25)

and $b \neq v_4$.

If $b \notin \{z_1, z_2\}$ and $b' \notin \{z_1', z_2'\}$, then since $\ell(C) > 8$, we have $b = v_3$ and $b' = v_{s-2} = v_4$. By symmetry, we may assume that $|L(v_4)| = 3$, and since $v_4 \in X'$, the choice of X' implies that $|L(v_5)| = 2$, $|L(v_6)| = 3$ and $|L(v_3)| = 2$. Consequently, $|L(v_2)| = 3$ and $|L(v_1)| = 2$. Let ψ be a coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2\}$ such that $\psi(v_4) \notin L(v_5)$; note that (5) implies that the vertex of $V(T) \setminus V(C)$ is not adjacent to a vertex of P, ensuring that such a coloring exists. Let $G' = G - V(T)$ and let L' be the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. The graph G' is not L'-colorable, and by (5) and (18), it satisfies (I) and (Q). This contradicts the minimality of G. Thus, we may assume that say $b \in \{z_1, z_2\}$.

If $b = z_1$, then (18) implies that $b \neq b'$ and $b' \in \{z'_1, z'_2\}$. Let $V(T) = \{b, b', t\}$, let $v' \in \{v_{s-1}, v_{s-3}\}$ be the neighbor of b' and let G_2 be the subgraph split off by $v_2 z_1 b' v'$. If $T \not\subseteq G_2$, then (11) implies that v_2 and v' have a common neighbor with list of size two, hence $v' = v_4 = v_{s-3}$ and $b' = z'_2$. By (5), we have $z'_2 = z_2$. Note that $t \neq z'_1$, since $b' \neq z'_1$. If t has a neighbor in P, then since z'_1 has degree at least three, (5) implies that $tp_0, z'_1 p_1 \in E(G)$. However, such a graph is L-colorable. It follows that t has no neighbor in P. Similarly, z_1 and z_2 have no neighbors in C other than v_2 and v_4 and no neighbor of v_7 is adjacent to a vertex of T. There exists an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(z_2)\}| \geq 2$. Let $G' = G - (V(T) \cup \{v_3, v_4, v_5\})$ with the list assignment L' such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_1), \psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$, and $L'(x) = L(x)$ otherwise. Observe that G' satisfies the assumptions of Theorem 2 and is not L'-colorable, contradicting the minimality of G.

Let us now consider the case that $T \subseteq G_2$. Since t has degree at least three, we conclude that the subgraph of G split off by the path $v_2 z_1 t b' v'$ is OBSTb1, $t = z_2$ and either $z'_2 = z_2$, $b' = z'_1$ and $s = 7$, or $b' = z'_2$ and $s = 9$. Suppose that b or b' has a neighbor in P. If $s = 7$, then the resulting graph would be L-colorable. If $s = 9$, then (5) implies that z'_1 has degree two. This is a contradiction, hence neither b nor b' has a neighbor in P. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(t)\}| \geq 2$. Let $G' = G - \{v_3, v_4, v_5, t\}$ if $s = 7$ and $G' = G - \{v_3, v_4, v_5, v_6, v_7, t\}$ if $s = 7$, with the list assignment L' such that $L'(x) = \{\psi(x)\}$ if $x \in \{v_1, v_2, z_1\}$, $L'(b') = \{\psi(b'), \psi(z_1)\}$ and $L'(x) = L(x)$ otherwise. Note that G' is not L'-colorable, thus it violates (OBSTb). Since b' and v_s are the only vertices with list of size two, G' contains OBSTb1a, OBSTb1b or OBSTb3 as a subgraph; and if $s = 9$, (5) implies that z'_1 belongs to this subgraph. However, in all the cases the resulting graph is L-colorable, which is a contradiction.

Therefore, we have $b = z_2$. Suppose that $b' \in V(C)$. If $b' = v_4$, then (25) implies that $v_4 \notin X$, thus (A5) holds and $v_5 \in V(T)$. This is a contradiction, as we would choose $b = v_5$. Therefore, $b' \neq v_4$, and (18) implies that the 2-chord $v_4 b b'$ splits off T, thus $b' = v_5$. Since $v_3 \notin V(T)$, we have $v_4 \notin X$ and (A5) holds by (25). However, since $|L(v_4)| = |L(v_5)| = 3$, we have $v_5 \notin X'$, and since $b' \in X'$, this is a contradiction.

Finally, consider the case that $b' \notin V(C)$. Note that $b' \neq z'_1$, since we already excluded the symmetric case $b = z_1$, hence $b' = z'_2$. Suppose first that $b = b'$. By (18), we have $v_{s-3} \in \{v_4, v_5\}$. If $v_{s-3} = v_4$, then let $V(T) = \{b, t, t'\}$, and note that $\{t, t'\} \cap \{z_1, z'_1\} = \emptyset$, by the choice of b and b'. Since z_1 and z'_1 have degree at least three, (5) implies that the vertices of T have no neighbors in P, and that the distance between T and $\{v_1, v_7\}$ is at least three. There exists an L-coloring ψ of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $|L(v_4) \setminus \{\psi(v_3), \psi(b)\}| \geq 2$. Let $G' = G - \{v_3, v_4, v_5, b, t, t'\}$ and L' the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$ and $L'(x) = L(x)$ otherwise. Observe that G' is not L'-colorable and satisfies (I). Since z_1 has degree at least three, (5) implies that G' satisfies (Q). It follows that G' contains a subgraph H isomorphic to OBSTb1 or OBSTb2. By (5), we have $z_1, v_7 \in V(H)$. If H is OBSTb1, then C has a 3-chord $v_2z_1xv_7$ contradicting (18). If H is OBSTb2, then G contains a path $v_2z_1xyzv_7$, where y has a neighbor in T. However, then t or t' has degree two by (5), which is a contradiction.

If $v_{s-3} = v_5$, then both X and X' satisfy (A5). By (18), we have $z_1 \neq z'_1$. Since both z_1 and z'_1 have degree at least three, (5) implies that b has no neighbor in P and is in distance at least three from $\{v_1, v_7\}$. Let ψ be an L-coloring of the subgraph of G induced by $V(P) \cup V(T) \cup \{v_1, v_2, v_3\}$ such that $\psi(v_5) \notin L(v_6)$. Let $G' = G - \{v_3, v_4, v_5, v_6, b\}$ and L' the list assignment such that $L'(v_1) = \{\psi(v_1)\}$, $L'(v_2) = \{\psi(v_2)\}$, $L'(x) = L(x) \setminus \{\psi(y)\}$ if x has a neighbor $y \in V(T)$ and $L'(x) = L(x)$ otherwise. Observe that G' is not L'-colorable and satisfies (I) and (Q). By the minimality of G, G' contains a subgraph H isomorphic to OBSTb1 or OBSTb2. The distance between the neighbors of b is at least three, thus at most one of them belongs to H and has list of size two. It follows that H is OBSTb1 and $v_7 \in V(H)$. However, then z_1 or z'_1 has degree two by (5), which is a contradiction.

We conclude that $b \neq b'$. Since T has two vertices that do not belong to C, neither X nor X' satisfies (A5). Since $v_3 \notin V(T)$, by (25) we have $v_4 \notin V(T)$, and symmetrically, $v_{s-3} \notin V(T)$; thus, $v_{s-3} \neq v_4$. Let $\{t\} = V(T) \setminus \{b, b'\}$. Consider the 3-chord $Q = v_4bb'v_{s-3}$ and the subgraph G_2 split off by it. If $T \not\subset G_2$, then (11) implies that v_4 and v_{s-3} have a common neighbor, and thus $s = 9$. If $T \subset G_2$, then we similarly conclude that $v_4btb'v_{s-3}$ splits off OBSTb1, i.e., $s = 11$ and t is adjacent to v_6.

Let $S_1 = L(v_2) \setminus (L(v_1) \setminus L(p_2))$ and $S_2 = L(v_{s-1}) \setminus (L(v_s) \setminus L(p_0))$. By the minimality of G, we have $|S_1| = |S_2| = 2$, as otherwise we can remove the edge v_1v_2 or $v_{s-1}v_s$. Suppose now that there exists an L-coloring ψ of T such that for every $c_1 \in S_1$ and $c_2 \in S_2$, there exists an L-coloring φ of the subgraph of G induced by $V(T)\cup\{v_2,v_3,\ldots,v_{s-1}\}$ such that $\varphi(v_2)=c_1$, $\varphi(v_{s-1})=c_2$ and $\varphi(x)=\psi(x)$ for $x\in V(T)$. Let $G'=G-(V(T)\cup\{v_3,v_4,\ldots,v_{s-2}\})$ and let L' be the list assignment such that $L'(x)=L(x)\setminus\{\psi(y)\}$ if x has a neighbor y in $V(T)$ and $L'(x)=L(x)$ otherwise. The choice of ψ implies that every L'-coloring of G' corresponds to an L-coloring of G, thus G' is not L'-colorable. Note that no vertex of T is adjacent to a vertex of P and that the distance between T and $\{v_1,v_s\}$ is at least three, since otherwise (5) would imply that z_1 or z'_1 has degree two. Thus, G' satisfies (S3) and (I). Furthermore, it satisfies (OBSTa), since otherwise a triangle of G' would be in distance at most 7 from T, contradicting $t(G)\geq B$. Therefore, G' would be a smaller counterexample to Theorem 2, which is a contradiction.

We conclude that no such L-coloring ψ exists. In particular, for any color $c\in L(b)$, the list $L(v_4)\setminus\{c\}$ has size two and intersects $L(v_3)$. It follows that $L(v_3)\subseteq L(v_4)=L(b)$, and symmetrically, $L(v_{s-2})\subseteq L(v_{s-3})=L(b')$. Similarly, we conclude that $L(v_3)=S_1$, $L(v_{s-2})=S_2$, $L(v_5)\subseteq L(v_4)$, $L(v_{s-4})\subseteq L(v_{s-3})$, and if $s=11$, then $L(v_5),L(v_7)\subseteq L(v_6)=L(t)$. If $L(v_3)=L(v_5)=S_1$, then choose $\psi(b)\in S_1$ arbitrarily. Now, regardless of the values of c_1, c_2 and the rest of ψ, we can choose the color of v_4 to be the unique color in $L(v_4)\setminus S_1$, and the L-coloring φ will exist. Therefore, $L(v_5)\neq S_1$ and $L(v_{s-4})\neq S_2$. Similarly, if $s=11$, then $L(v_5)\neq L(v_7)$. Let $\{c_3\}=L(v_5)\cap S_1$. Let $\psi(b)$ be the unique color in $S_1\setminus L(v_5)$. Furthermore, if $s=11$ then let $\psi(t)=c_3$, and if $s=9$ then let $\psi(b')=c_3$. Observe that ψ (extended to the third vertex of T arbitrarily) has the required property—if $c_1\neq\psi(b)$, then we can color v_3 by $\psi(b)$, so that two neighbors of v_4 have the same color. And if $c_1=\psi(b)$, then we can color v_3 by c_3, v_4 by the color in $L(v_4)\setminus S_1$ and v_5 with c_3, so that v_6 has two neighbors with the same color. This contradiction finishes the proof of Theorem 2. □

3 Concluding remarks

The proof of Theorem 2 follows the lines of the original Thomassen's proof [11]. However, a basically unavoidable part of the proof—the need to handle 2-chords, so that we can color and remove a 5-face in (21)—forces us to deal with a large number of counterexamples to the claim "every precoloring of a path of length two can be extended." Especially painful is the obstruction OBSTx1, which even applies to a path of length one. One could ask whether we could not avoid this by forbidding vertices with list of size two in triangles completely. This cuts down the number of obstructions significantly, and indeed, this was our original aim. However, at the final stage of the proof, we would only end up knowing that there is a triangle whose distance is at most two from a vertex on each side of the precolored path P. This is a quite small amount of structure to work with, making the arising case analysis extremely difficult. Additionally, one runs into trouble if these two vertices are in fact identical, which would essentially force extending Corollary 3 to precolored cycles of length at most 10. The number of obstacles for such cycles then becomes rather large, and it is not quite clear how such an extension of Corollary 3 could be proved.

Another point where one could hope to save on obstructions is by only considering the precoloring of a path of length at most 4 in case that (≤ 4)-cycles are far enough from it. However, there are many places throughout the proof where it is useful to extend the coloring of a path of length two to a coloring of a path of length five, and it is unclear how to handle these situations using only paths of length four.

Consequently, we end up with a nontrivial number of obstructions, and the proof becomes rather technical. Despite the length of this paper, still a large amount of work is hidden in the need to carefully verify all the claims; in particular, we in general do not give detailed proofs of colorability of the described graphs. We feel that doubling the length of the paper by spelling out all these technical details would not make the exposition any clearer or more believable. Similar remarks apply to other results proved using this technique (even the original paper of Thomassen [6], although written quite shortly, becomes rather long when all details are worked out). Given the rather repetitive nature of the arguments, one wonders whether it would not be possible to apply computer to obtain such proofs. Let us however note that many of the reductions appearing in our proof are quite tricky and it is not immediately obvious how they could be obtained mechanically.

On the positive side, Theorem 2 is somewhat interesting even for graphs of girth five, since it describes which precolorings of a path of length at most five can be extended. This might be useful as a technical tool in further study of 4-critical graphs of girth five. Similarly, Theorem 2 and Corollary 3 give interesting information regarding graphs with exactly one cycle of length at most four.

Compared with the solution to Havel's problem [6], our proof is rather elementary, not using any deeper results. Would it be possible to apply the techniques of [6] instead? Possibly, but it would require developing a new proof of 3-choosability of planar graphs of girth 5 based on reducible configurations and discharging. While our initial inquiry in that direction was somewhat encouraging, it seems inevitable that the set of reducible configurations needed would be rather large (possibly hundreds as opposed to about 10 needed in [6] for the case of 3-coloring), so the proof would become of somewhat dubious value.

Finally, let us remark that we could require a much weaker assumption on the distance between 4-cycles, since in most of the arguments only triangles cause problems. However, for obvious reasons we did not want to complicate the proof any more.

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