Bipartizing fullerenes*

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Abstract

A fullerene graph is a cubic bridgeless planar graph with twelve 5-faces such that all other faces are 6-faces. We show that any fullerene graph on n vertices can be bipartized by removing $O(\sqrt{n})$ edges. This bound is asymptotically optimal.

Keywords: Fullerene graph; Fullerene stability; Bipartite spanning subgraph

1 Introduction

Fullerenes are carbon-cage molecules comprised of carbon atoms that are arranged on a sphere with pentagonal and hexagonal faces. The icosahedral C_{60}, well-known as Buckminsterfullerene was found by Kroto et al. [10], and later confirmed by experiments by Krätchmer et al. [9] and Taylor et al. [12]. Since the discovery of the first fullerene molecule,

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the fullerenes have been objects of interest to scientists all over the world.

From the graph theoretical point of view, the fullerenes can be viewed as cubic 3-connected graphs embedded into a sphere with face lengths being 5 or 6. Euler's formula implies that each fullerene contains exactly twelve pentagons, but provides no restriction on the number of hexagons. In fact, it is not difficult to see that mathematical models of fullerenes with precisely α hexagons exist for all values of α with the sole exception of $\alpha = 1$. See [3, 5, 6, 11] for more information on chemical, physical, and mathematical properties of fullerenes.

The question of stability of fullerene molecules receives a lot of attention. The goal is to obtain a graph-theoretical property whose value influences the stability. Different properties, like the number of perfect matchings [7] or the independence number [4] were considered. The property investigated in this paper is how far the graph is from being a bipartite graph, which was suggested by Došlić [1] and further considered in [2]. Despite of the effort none of the so far considered parameters works in all cases. Hence more research is still needed.

For a plane graph H, let $F(H)$ be the set of the faces of H. Let H be a fullerene graph, and let K_H be the weighted complete graph whose vertices correspond to the 5-faces of H and the weight of the edge joining two 5-faces f_1 and f_2 is equal to the distance from f_1 to f_2 in the dual of H. Let $b(H)$ be the size of the minimum set $S \subseteq E(H)$ such that $H - S$ is bipartite. Došlić and Vukičević [2] proved the following:

Theorem 1. *If H is a fullerene graph, then $b(H)$ is equal to the minimum weight of a perfect matching in K_H.*

A corollary of the above theorem is a polynomial-time algorithm for finding a set of edges S whose removal makes the graph bipartite.

Došlić and Vukičević [2] conjectured that $b(H) = O(\sqrt{|V(H)|})$. In fact, they gave the following stronger conjecture.

Conjecture 2. *If H is a fullerene graph with n vertices, then $b(H) \leq \sqrt{12n/5}$.*

The main result of this paper is an upper bound on $b(H)$, confirming the weaker version of the conjecture.

Theorem 3. *If H is a fullerene graph with n vertices, then $b(H) = O(\sqrt{n})$.*

2 Proof of Theorem 3

Let H be a fullerene graph. A *patch with boundary o* is a 2-connected subgraph $G \subseteq H$ such that $o \in F(G)$ (usually, we consider o to be the outer face of G) and $F(G) \setminus F(H) \subseteq \{o\}$ (but it is possible for the boundary o to be also a face of G). Let v be a vertex incident with o. If $\deg_G(v) = 3$, then v is a 3-*vertex (with respect to o)*, otherwise v is a 2-*vertex (with respect to o)*. An edge e incident with o is a 22-*edge* (resp. *a* 33-*edge*) if both vertices incident with e are 2-vertices (resp. 3-vertices) with respect to o. If e is neither a 22-edge nor a 33-edge, then it is a 23-*edge*. The *description* $D(o)$ of the boundary o is the cyclic sequence in that A represents a 33-edge, B represents a 22-edge, and a maximal consecutive segment of 23-edges is represented by the integer giving its length. For example, the boundary of the patch consisting of a 5-face and a 6-face sharing an edge is described as $BB2BBB2$.

Let $s(o)$ and $t(o)$ be the numbers of 22-edges and 33-edges of o, respectively, and let $s_2(o)$ be the number of pairs of consecutive 22-edges of o. Let $p(G)$ be the number of 5-faces of G distinct from o. The following lemma relates the number of 22- and 33-edges; a similar relation was derived by Kardoš and Škrekovski [8].

Lemma 4. *If G is a patch with the boundary o, then $s(o) = 6 - p(G) + t(o)$.*

Proof. Suppose that the length of o is ℓ. Let $n = |V(G)|$, $m = |E(G)|$ and let f be the number of faces of G. Since each edge of G is incident with two faces,

$$2m = 6(f - p(G) - 1) + 5p(G) + \ell,$$

i.e., $\ell = 2m + p(G) + 6 - 6f$. Note that the number of 2-vertices is $(\ell + s(o) - t(o))/2$, which can be easily seen from the modification of the boundary by adding $s(o)$ and deleting $t(o)$ 3-vertices so that there is no 33-edge or 22-edge. Thus

$$2m = 3n - (\ell + s(o) - t(o))/2.$$

Substituting for ℓ, we obtain

$$3m = 3n + 3f - 6 + \frac{6 - p(G) + t(o) - s(o)}{2}.$$

By Euler's formula, $m = n + f - 2$, thus $6 - p(G) + t(o) - s(o) = 0$ and the claim of the lemma follows. □

A patch G with the boundary o is a *fat worm* if $p(G) = 0$, the subgraph of G induced by $V(G) \setminus V(o)$ is a path P, and the edges of $E(G) \setminus E(P)$ incident with each two consecutive inner vertices of P are not incident to a common face of G. See Figure 1(a). Note that in this case, the description of o is

- $BB2B(2k+2)BB2B(2k+2)$ if P has length $2k+1$ and
- $BB2B(2k+2)B2BB(2k+4)$ if P has length $2k+2$.

We consider the patch with exactly one vertex not incident with o (and boundary $BB2BB2BB2$) to be a fat worm as well (in this case, P has length 0). The patch G is a *slim worm* if $p(G) = 0$, $V(G) = V(o)$ and $t(o) = 0$. Geometrically, it is a straight line of hexagons, see Figure 1(b). Note that $D(o) = BBB(2k)BBB(2k)$ for some k (or $D(o) = BBBBBB$, when $k = 0$ and o is a 6-face). The patch G is a *worm* if it is a fat worm or a slim worm. The *shell* is the patch G with boundary o such that $p(G) = 0$ and $D(o) = BB4BB4BB4$ (having 4 internal vertices). See Figure 1(c).

An ℓ-*chord* of a cycle C in a patch G is a path of length ℓ with distinct endvertices belonging to $V(C)$ such that the inner vertices and edges of the path do not belong to C. We say a chord instead of

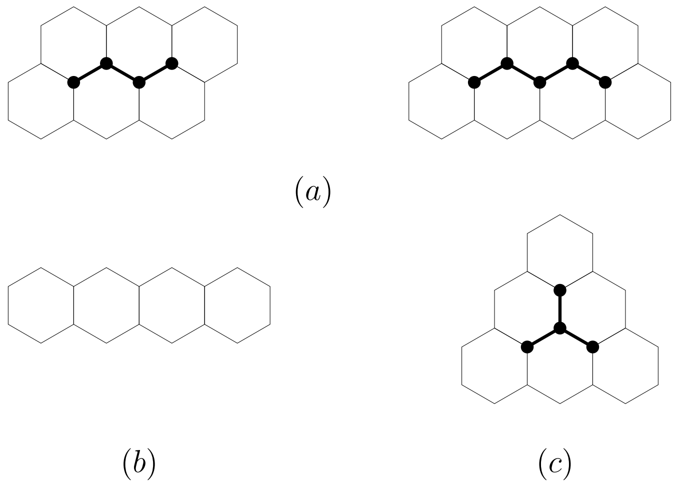

Figure 1: A fat worm, a slim worm and the shell.

a 1-chord. Consider an ℓ-chord Q of the boundary o of a patch G. Let G_1 and G_2 be the two patches into that Q splits G (i.e., the subgraphs such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = Q$ and $G_1 \neq Q \neq G_2$), and o_1 and o_2 their boundaries. We say that Q *splits off a face* if $G_1 = o_1$ or $G_2 = o_2$. The patch G is *decomposable* if it contains a *simplifying cut*, that is

- an ℓ-chord Q of o with $\ell \leq 3$ such that $t(o_1) + t(o_2) < t(o)$, or
- two 4-chords $Q_1 = v_0v_1v_2v_3v_4$ and $Q_2 = w_0w_1w_2w_3w_4$ such that v_0w_0, v_2w_2 and v_4w_4 are edges of G. See Figure 2.

Otherwise, we call G *indecomposable*. We say that G is a *normal patch* if G is indecomposable, no 5-face of G distinct from o shares an edge with o and G is neither a worm nor a shell.

Lemma 5. *Let G be a normal patch with boundary o and Q an ℓ-chord of o, with $\ell \leq 3$. Then $\ell \geq 2$ and Q splits off a face. Furthermore, the number of 33-edges incident with the endvertices of Q is most $\ell - 2$.*

Proof. Let G_1 and G_2 with boundaries o_1 and o_2, respectively, be the patches to that Q splits G. Let

$$Q = q_0q_1 \ldots q_\ell \quad \text{and} \quad o_2 = q_0v_1v_2 \ldots v_aq_\ell q_{\ell-1} \ldots q_0.$$

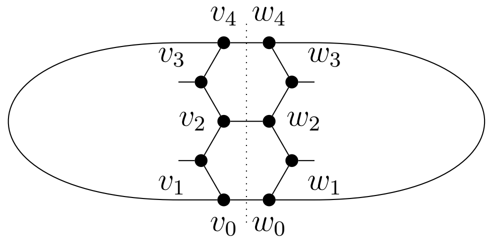

Figure 2: Two 4-chords.

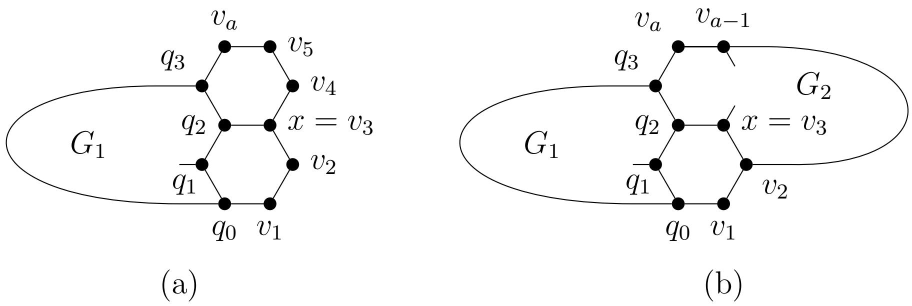

Figure 3: 3-chords from Lemma 5.

Suppose first that $\ell = 1$. Since G is not a slim worm, there exists an edge $e \in E(G)$ that either is a 33-edge of o or is incident with a vertex in $V(G) \setminus V(o)$. Let us choose the chord Q and the patches G_1 and G_2 so that $e \in E(G_2)$ and G_2 is minimal. As G is indecomposable, each 33-edge of G is also a 33-edge in G_1 or G_2. It follows that v_1 and v_a are 2-vertices, and since the internal face incident with q_0v_1 has length six, v_2 and v_{a-1} must be adjacent. Since $e \in E(G_2)$, we have $G_2 \neq o_2$; hence, v_2v_{a-1} is a chord of o. The chord v_2v_{a-1} splits G to patches G'_1 and G'_2 with $G'_2 \subset G_2$. However, this contradicts the choice of Q, since it is easy to see that $e \in E(G')$. We conclude that o is an induced cycle.

Suppose now that $\ell = 2$. By symmetry between G_1 and G_2, we may assume that q_1 is a 2-vertex in G_2. Since $t(o_1) + t(o_2) \geq t(o)$, we have that v_1 and v_a are 2-vertices, and it follows that $G_2 = o_2$ is a face split off by Q.

Finally, suppose that $\ell = 3$. Suppose first that both q_1 and q_2 are 2-vertices in G_2, and thus q_1q_2 is a 33-edge with respect to o_1. Since $t(o_1) + t(o_2) \geq t(o)$, at least one of v_1 and v_a (say v_1) is a 2-vertex. Thus, $V(Q) \cup \{v_1, v_2, v_a\}$ are all incident with a common face, which is only possible if $v_2 = v_a$ and G_2 consists of a single face. It follows that Q splits off a face.

The case that both q_1 and q_2 are 2-vertices in G_1 is symmetrical. Hence, without loss of generality, we assume that q_1 is a 2-vertex and q_2 is a 3-vertex in G_2. As $t(o_1)+t(o_2) \geq t(o)$, we infer that both v_1 and v_a are 2-vertices. Let $x \notin \{q_1, q_3\}$ be the third neighbor of q_2. Observe that

- if $x \in V(o)$, then both xq_2q_3 and $xq_2q_1q_0$ split off a face (for the former, note that the edge joining v_{a-1} with a neighbor of x is not a chord, since we already proved that o is an induced cycle). See Figure 3(a).
- if $x \notin V(o)$, then x and v_2 are adjacent, $v_1v_2xq_2q_1q_0$ is a face and we may apply the same observations to the 3-chord $v_2xq_2q_3$. See Figure 3(b).

By symmetry, this argument also holds for o_1. Hence by repeating the argument we conclude that G is a fat worm, contradicting the assumption that G is a normal patch.

Furthermore, note that if Q splits off a face, then $t(o) = t(o_1) + t(o_2) - (\ell - 2) + k$, where k is the number of 33-edges incident with q_0 or q_ℓ. Since G is indecomposable, it follows that $k \leq \ell - 2$. □

For a patch G with boundary o, let $G' \subseteq G$ be the subgraph consisting of the outer layer of the faces of G; that is, e is an edge of G' if and only if it is incident with a face that shares an edge with o. Let $S \subseteq V(G) \setminus V(o)$ be the set of vertices that have at least two neighbors in o. Let $o' = G' - (V(o) \cup S)$. See Figure 4(a).

Lemma 6. *If G is a normal patch with boundary o, then o' is a cycle, and the patch bounded by o' satisfies $t(o') = t(o)$, $s(o') = s(o)$ and $s_2(o') \geq s_2(o)$. Furthermore, $\ell(o') = \ell(o) + 2p(G) - 12 - 2s_2(o)$.*

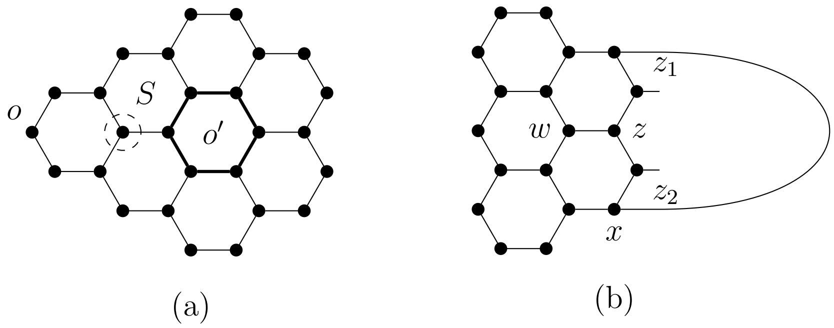

Figure 4: Patch G and o' and a configuration from Lemma 6.

Proof. Since G is not a fat worm, we have $|V(G) \setminus V(o)| > 1$. If two vertices of S were adjacent, then $|V(G) \setminus V(o)| = 2$ by Lemma 5 and G would be a fat worm, thus S is an independent set and o' is not empty. Lemma 5 also implies that $G - V(o)$ is connected, and since S only contains vertices whose degree in $G - V(o)$ is one, o' is connected as well.

Suppose that a vertex w of o' is adjacent to more than one vertex of S. Since G is not the shell, w is adjacent to exactly two vertices in S; let z be the neighbor of w not in S. Since G is not a fat worm, we have $z \notin V(o)$. Let z_1 and z_2 be the neighbors of z distinct from w; since $z \notin S$, we may assume that $z_2 \notin V(o)$. Let f be the face of G incident with w, z and z_2, and let x be the neighbor of z_2 in f distinct from z. Note that f is incident with a neighbor of w that belongs to S, and thus f shares an edge with o. Hence f is a 6-face and we have $x \in V(o)$. If $z_1 \in V(o)$, then the 3-chord $z_1 z z_2 x$ contradicts Lemma 5. Otherwise, by a symmetric argument we conclude that a face f' incident with w, z and z_1 is also a 6-face sharing an edge with o, see Figure 4(b). However, $f \cup f'$ forms a simplifying cut (a pair of 4-chords) in G, which is a contradiction. Therefore, each vertex of o' has at most one neighbor in S.

By Lemma 5, no vertex of o' has a neighbor both in S and in o, since at least one of the two resulting 3-chords would not split off a face. If v is a vertex of o' that has a neighbor in o or S, then v has two neighbors in o', and thus o' has at least three vertices.

Suppose that o' contains a bridge $e = uv$. Note that both faces f_1 and f_2 of G incident with e share an edge with o. As $u, v \notin S$, these two vertices do not lie on 2-chords. Note that $f_1 \cup f_2$ contains an ℓ_u-chord P_u of o such that $u \in V(P_u)$ and $v \notin V(P_u)$, where $3 \leq \ell_u \leq 5$. Similarly, let P_v be an ℓ_v-chord of o such that $v \in V(P_v)$ and $u \notin V(P_v)$. As neither P_u nor P_v splits off a face, Lemma 5 implies that $\ell_u, \ell_v \geq 4$. Since f_1 and f_2 are 6-faces, we conclude that P_u and P_v are 4-chords. Lemma 5 further implies that u and v are middle vertices of P_u and P_v, thus f_1 and f_2 is a pair of 4-chords forming a simplifying cut. This is a contradiction; therefore, o' is 2-edge-connected. Since $o' \subset G'$, every edge of o' is incident with a face that shares an edge with o. We conclude that o' is a cycle.

Consider now a 33-edge x_1x_2 in o and let $x_1x_2x_3x_4x_5x_6$ be the incident 6-face. Lemma 5 implies that each of x_3 and x_6 has only one neighbor in o, as otherwise one of them would belong to a 2-chord whose endpoint is incident with a 33-edge x_1x_2. Therefore, $x_3, x_6 \notin S$ and $x_3x_4x_5x_6$ is a part of o', and x_4x_5 is a 33-edge with respect to o'. It follows that $t(o') \geq t(o)$. On the other hand, consider a 33-edge y_4y_5 of o', and let $y_3y_4y_5y_6$ be a part of the boundary of o'. As y_4 and y_5 are 3-vertices in o', there exists a 6-face $y_1y_2y_3y_4y_5y_6$ in G, and y_1y_2 is a 33-edge in o. Hence, we have $t(o') = t(o)$ and by Lemma 4, $s(o') = s(o)$.

Similarly, consider a part $z_0z_1z_2z_3z_4z_5z_6$ of o, where z_2z_3 and z_3z_4 are 22-edges. The common neighbor z of z_1 and z_5 belongs to S, and its neighbor z' distinct from z_1 and z_5 belongs to o'. As we observed before, both neighbors z'_1 and z'_2 of z' distinct from z belong to o'. Furthermore, by Lemma 5, the endpoints of the 2-chord z_1zz_5 are incident with no 33-edges, thus z_0 and z_6 are 2-vertices. It follows that both z'_1 and z'_2 have a neighbor in o, and z'_1z' and z'_2z' are 22-edges with respect to o'. Hence, we conclude that $s_2(o') \geq s_2(o)$.

In fact, $D(o')$ can be obtained from $D(o)$ in the following way: Add 0 between each two consecutive letters in $D(o)$. Since endvertices of a 2-chord of o are not incident with 33-edges, if $B0B$ appears in the resulting sequence, then it is as a part of a subsequence n_1B0Bn_2, where $n_1, n_2 \geq 3$. We construct $D(o')$ by

- for each $n_1 B 0 B n_2$ subsequence, decreasing each of n_1 and n_2 by 3,
- for each B not contained in such a subsequence, decreasing each of the neighboring integers by 1,
- for each A, increasing each of the neighboring integers by 1, and
- suppressing any zeros.

Note that the increases/decreases are cumulative, e.g., if $D(o)$ contains a subsequence $A3B2B$, then the sequence $D(o')$ contains a subsequence $A3B0B$ (or $A3BB$ after suppressing zeros). By Lemma 4, $t(o) - s(o) = p(G) - 6$, and the formula for the length of o' follows:

$$\begin{aligned}\ell(o') &= \ell(o) + 2t(o) - 2(s(o) - 2s_2(o)) - 6s_2(o) \\ &= \ell(o) + 2p(G) - 12 - 2s_2(o).\end{aligned}$$

□

Consider a patch G with boundary o_1. A sequence of cycles o_1, o_2, ..., o_k (with $k \geq 2$) is called an *uninterrupted peeling* if for $1 \leq i < k$, the subpatch of G bounded by o_i is normal and $o_{i+1} = o_i'$.

Lemma 7. *Let o be the boundary of a patch G such that $p(G) \neq 6$. If $o = o_1$, o_2, ..., o_k is an uninterrupted peeling, then the number of vertices of G outside of (and not including) o_k is at least $4k^2/9$.*

Proof. By Lemma 6, we have $s_2(o_1) \leq s_2(o_2) \leq \ldots \leq s_2(o_k)$. Moreover, Lemma 6 also implies that the sequence $\ell(o_1), \ldots, \ell(o_k)$ is concave.

Let a be the largest index such that $\ell(o_1) < \ldots < \ell(o_a)$ and let b be the smallest index such that $\ell(o_b) > \ldots > \ell(o_k)$. Note that if the whole sequence is decreasing then $a = b = 1$ and similarly if the whole sequence is increasing then $a = b = k$, hence $a \leq b$ in all the cases. We compute a lower bound on the the number of vertices of G outside of o_k as

$$\sum_{i=1}^{k-1} \ell(o_i) = \sum_{i=1}^{a-1} \ell(o_i) + \sum_{i=a}^{b-1} \ell(o_i) + \sum_{i=b}^{k-1} \ell(o_i).$$

First, we deal with the middle term. Let $m = b - a$. Suppose that $a < b$. In this case, we have $\ell(o_a) = \ell(o_{a+1}) = \ldots = \ell(o_b)$; let $r = \ell(o_a)$. By Lemma 6, $s_2(o_i) = p(G) - 6$ for $a \leq i < b$. Since $p(G) \neq 6$, we conclude that $s_2(o_a) \geq 1$. It follows that $D(o_a)$ contains a subsequence $n_1 BB n_2$, where $n_1, n_2 \geq 3$ by Lemma 5. By Lemma 4, $t(o_a) - s(o_a) = p(G) - 6 = s_2(o_a)$. As $s(o_a) \geq 2s_2(o_a)$, we conclude that $t(o_a) \geq 3$, and thus $n_1 + n_2 + 5 \leq r$. By symmetry, assume that $2n_1 \leq r - 5$. As observed in the proof of Lemma 6, $D(o_{a+1})$ contains a subsequence $n'_1 BB n'_2$, where $n'_1 \leq n_1 - 2$ (the equality is achieved if n_1 is adjacent to A in $D(o_a)$). The same observation applies to o_{a+1}, $\ldots$, o_{b-2}. In the normal patch o_{b-1}, the integers adjacent to BB are greater or equal to three, thus $n_1 \geq 2m + 1$ and $r \geq 4m + 7$. It follows that $\sum_{i=a}^{b-1} \ell(o_i) = mr \geq m(4m + 7)$. In the case that $a = b$, we have $\sum_{i=a}^{b-1} \ell(o_i) = 0 = m(4m + 7)$, since $m = 0$.

Now we deal with the other terms of the sum. If $a > 1$, then the sequence $\ell(o_1), \ell(o_2), \ldots, \ell(o_{a-1})$ dominates the arithmetic sequence with the first element $\ell(o_1) \geq 5$ and step 2 due to Lemma 6 and the fact that $p(G) - 6 - s_2(o_i) \geq 1$ for $1 \leq i \leq a - 1$. Hence $\sum_{i=1}^{a-1} \ell(o_i) \geq \sum_{i=1}^{a-1}(3+2i) = (a-1)(a+3)$. If $a = 1$ then $\sum_{i=1}^{a-1} \ell(o_i) = 0 = (a-1)(a+3)$.

Similarly, the sequence $\ell(o_b), \ell(o_{b+1}), \ldots, \ell(o_{k-1})$ dominates the arithmetic sequence with the last element $\ell(o_{k-1}) \geq 7$ and step -2, hence $\sum_{i=b}^{k-1} \ell(o_i) \geq \sum_{i=1}^{k-b}(5 + 2i) = (k - b)(k - b + 6)$.

Note that $(a-1)+m+(k-b) = k-1$. Summing these inequalities, we obtain

$$\begin{aligned}\sum_{i=1}^{k-1} \ell(o_i) &\geq (a-1)(a+3) + m(4m+7) + (k-b)(k-b+6) \\ &\geq (a-1)^2 + 4m^2 + (k-b)(k-b+2) + 1 \\ &\geq 4k^2/9,\end{aligned}$$

where the lower bound in the last inequality is achieved for $a - 1 = 4k/9$, $k - b = 4k/9 - 1$ and $m = k/9$. Since all the cycles $o_1, \ldots, o_{k-1}$ are strictly outside of o_k, the claim follows. □

Lemma 8. *Let H be a fullerene with n vertices and f a 5-face of H. There exist at least five 5-faces distinct from f whose distance to f in the dual of H is at most $\sqrt{63n/2} + 14$.*

Proof. We define a rooted tree T with each vertex of T corresponding to a patch $G \subseteq H$ such that $p(G) \neq 0$ and $p(G) \neq 6$. Furthermore, we assign a weight $d(e)$ to each edge e of T. The root of T is the patch $G_0 = H$ whose boundary is the cycle bounding f, i.e., $p(G_0) = 11$. Suppose that a patch G with boundary o is a vertex of T. Let us note that G is neither a worm nor the shell, since $p(G) > 0$. The sons of G in the tree are defined as follows:

(a) If $p(G) \in \{1, 7\}$ and o shares an edge with a 5-face of G, then G is a leaf of T.

(b) If G is a normal patch, then G has a single son G', equal to the last element of the maximal uninterrupted peeling starting with o. The weight of the the edge e joining G with G' is equal to the length (number of patches) of the uninterrupted peeling. Note that G' is not a normal patch.

(c) If G has a simplifying cut, then let o_1 and o_2 be the boundaries of the two patches G_1 and G_2 to that it splits G. Note that $t(o_1) + t(o_2) < t(o)$. The patch G_i (with $i \in \{1, 2\}$) is a son of

G if $p(G_i) \neq 0$ and $p(G_i) \neq 6$. In that case, the edge between G and G_i has weight 1. Since $0 < p(G) < 12$ and $p(G) \neq 6$, G has at least one son.

(d) Finally, if G is indecomposable, $p(G) \notin \{1, 7\}$ and o shares an edge with a 5-face f', note that there exists an ℓ-chord (with $\ell \leq 4$) splitting off f' (otherwise f' would be incident with a chord and a 2-chord and both of them would witness the decomposability of G). We let the son G' of G with boundary o' be the patch obtained from G by removing edges incident to both f' and o and by removing isolated vertices. We let the edge of T between G and G' have weight 1.

The *type* of G is defined according to the rule ((a) to (d)) in that its sons are described.

Observe that at least five 5-faces distinct from f share edges with boundaries of the patches forming the vertices of T of type (a) or (d). Indeed, either all 5-faces are reachable in this way, or there are exactly six potentially unreachable 5-faces contained in a single patch that is a leaf of T, or split off by a simplifying cut from an internal vertex of T. Let T_1 be a subtree of T of smallest possible depth that contains five vertices of type (a) or (d). We choose T_1 to be minimal, i.e., all leaves of T_1 are of type (a) or (d).

Consider a vertex G_1 with a son G_2 in T_1, and let o_1 and o_2 be the boundaries of these patches. If G_1 is of type (b), then $p(G_1) = p(G_2)$ and $t(o_1) = t(o_2)$ by Lemma 6. If G_1 is of type (c), then $p(G_1) \geq p(G_2)$ and $t(o_1) > t(o_2)$. If G_1 is of type (d), then $p(G_1) > p(G_2)$ and $t(o_1) \geq t(o_2) - 1$.

Let $P = p_1 p_2 \dots p_m$ be the path in T joining the root $G_0 = p_1$ with a leaf $G = p_m$ whose boundary is incident with a 5-face f'. Observe that the distance between f and f' in the dual of G is at most the sum of the weights of the edges of P, plus 1. Let o_0 and o be the boundaries of G_0 and G, respectively. Let m_b, m_c and m_d be the numbers of vertices of types (b), (c) and (d) in P distinct from G, respectively. By the observations in the previous paragraph, we have

$t(o) \leq t(o_0) + m_d - m_c = 5 + m_d - m_c$. By the choice of T_1, we have $m_d \leq 4$, and since $t(o)$ is nonnegative, $m_c \leq 9$. Therefore,

$$\sum_{p_i \text{ is non-normal}} d(p_i p_{i+1}) \leq m_c + m_d \leq 13.$$

Let $d_1, d_2, \ldots, d_{m_b}$ be the sequence of the weights of all edges $p_i p_{i+1}$ of P such that p_i is a normal patch; by the construction of T, p_{i+1} is not a normal patch in this case, hence $m_b \leq m_c + m_d + 1 \leq 14$. Using Lemma 7, we obtain

$$n \geq \frac{4}{9} \sum_{i=1}^{m_b} d_i^2 \geq \frac{4}{9m_b} \left(\sum_{i=1}^{m_b} d_i \right)^2 .$$

Therefore, the total weight of these edges is at most $\sqrt{63n/2}$, and the distance between f and f' is at most $\sqrt{63n/2} + 14$. □

Lemma 9. *Every graph G on 12 vertices with minimum degree 5 such that $K_{5,7} \not\subseteq G$ has a perfect matching.*

Proof. If G does not have a perfect matching, then there exists a set $S \subseteq V(G)$ such that $G - S$ has more than $|S|$ components of odd size. Consider such a set S, and observe that $|S| < 6$. As $\delta(G) \geq 5$, G is either $2K_6$ (and thus has a perfect matching) or G is connected. Therefore, $|S| \geq 1$.

If $|S| < 5$, then since $\delta(G) \geq 5$, no component of $G - S$ may consist of a single vertex, and hence $G - S$ has at most three odd components and $|S| \leq 2$. Since $\delta(G) \geq 5$, each component of $G - S$ has size at least 4. However, $G - S$ must have at least two components of odd size, thus it would have exactly two components of size 5. However, then $|S| = 2$, which is not smaller than the number of odd components.

Therefore, $|S| = 5$ and $G - S$ has at least 6 components of odd size. However, this is only possible if each component of $G - S$ consists of a single vertex, and hence $K_{5,7} \subseteq G$. □

Proof of Theorem 3. Let K'_H be the subgraph of K_H consisting of edges with weight at most $\sqrt{63n/2} + 14$. By Lemma 8, $\delta(K'_H) \geq 5$, and thus K'_H either has a perfect matching or $K_{5,7}$ as a subgraph, by Lemma 9. In the former case, the weight of each perfect matching in K'_H (and thus of the minimum-weight perfect matching in K_H) is at most $6(\sqrt{63n/2} + 14) = \sqrt{1134n} + 84$. In the latter case, note that the weights in K_H satisfy the triangle inequality, thus the weight of any edge in K_H is at most $2(\sqrt{63n/2} + 14)$, and we conclude that K_H has a perfect matching of weight at most $(5+2)(\sqrt{63n/2}+14) = \sqrt{3087n/2} + 98$. By Theorem 1, $b(H) = O(\sqrt{n})$. □

The multiplicative constant $\sqrt{3087/2} \approx 39.29$ is likely to be far from the best possible. Indeed, it can be somewhat improved by a more complicated analysis of our argument (e.g., observing that not all 5-faces can appear in T on the lowest possible level, indicating that some of the edges of K_H are much shorter than we estimated). Nevertheless, we could not improve it enough to approach the best known lower bound of $\sqrt{12/5} \approx 1.549$ of Došlić and Vukičević [2].

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