Maximum edge-cuts in cubic graphs with large girth and in random cubic graphs

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Abstract

We show that for every cubic graph G with sufficiently large girth there exists a probability distribution on edge-cuts of G such that each edge is in a randomly chosen cut with probability at least 0.88672. This implies that G contains an edge-cut of size at least $1.33008n$, where n is the number of vertices of G, and has fractional cut covering number at most 1.12776. The lower bound on the size of maximum edge-cut also applies to random cubic graphs. Specifically, a random n-vertex cubic graph a.a.s. contains an edge cut of size $1.33008n$.

1 Introduction

An *edge-cut* in a graph $G = (V, E)$ defined by $X \subseteq V$ is the set of edges with exactly one end vertex in X (and exactly one end vertex in $V \setminus X$). A *maximum edge-cut* is an edge-cut with the maximum number of edges. The size of a maximum edge-cut is an important graph parameter intensively studied both in structural and algorithmic graph theory. For example in algorithmic theory, it attracted a lot of attention because of an approximation algorithm based on the semidefinite programming by Goemans and

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Williamson [3] which is the best possible under reasonable computational complexity assumptions [8]. In this paper, we provide new structural results on maximum cuts in cubic graphs, i.e., graphs with all vertices of degree three.

We prove a new lower bound on the size of a maximum edge-cut in a cubic graph with no short cycle and in a random cubic graph. Let us now mention some earlier results. In 1990, Zýka [17] proved that the size of the maximum edge-cut in cubic graphs with large girth is at least $9n/7 - o(n) = 1.28571n - o(n)$. A better bound $1.3056n$ can be obtained from a recent result [7] on independent sets in cubic graphs with large girth. The asymptotic lower bound for a maximum edge-cut in random cubic graphs of $1.32595n$ was given by Díaz, Do, Serna and Wormald [1]. The experimental evidence suggests that almost all n-vertex cubic graphs contain an edge-cut of size at least $1.382n$ [15]. On the other hand, the best known upper bound is $0.9351m = 1.4026n$ which applies both to random cubic graphs and cubic graphs with large girth. The upper bound was first announced by McKay [9], its rigorous proof can be found in [4]. The problem could also be translated to a problem in statistical physics and applying non-rigorous methods suggests that the size of a maximum edge-cut for almost all n-vertex graphs is at most $1.386n$ [16].

The problems of determining the size of a maximum edge-cut in random cubic graphs (more generally in random regular graphs) and in cubic (regular) graphs with large girth are closely related. On one hand, Wormald showed in [13] that a random cubic graph asymptotically almost surely (a.a.s.) contains only $o(n)$ cycles shorter than a fixed integer g. Therefore, we can a.a.s. remove a small number (which means $o(n)$) of vertices to obtain a subgraph with large girth and only $o(n)$ vertices of degree less than three.

On the other hand, Hoppen and Wormald [6] have recently developed a technique for translating many results for random r-regular graphs to r-regular graphs with sufficiently large girth. In particular, they are able to translate bounds obtained by analyzing the performance of so-called locally greedy algorithms for a random regular graphs. These algorithms and their analysis provide the currently best known asymptotic bounds to many parameters of random regular graphs, for example an upper bound on the size of the smallest dominating set [2]. The main tool for the analysis of such algorithms as well as for analysis of many other random processes is the *differential equation method* developed by Wormald [14].

Bounds on maximum edge-cuts are closely related to the concept of fractional cut coverings. A *fractional cut covering* of a graph G is a parameter analogous to a fractional coloring of G. It was first introduced by Šámal [10] under the name cubical colorings and he also related this parameter to graph homomorphisms. These ideas are further developed in [11, 12]. The aim is to assign non-negative weights to edge-cuts in G in such a way that for each edge e of G the sum of weights of the cuts containing e is at least one. The *fractional cut covering number* is the minimum sum of weights of cuts forming a fractional cut covering. Our approach in this paper gives also an upper bound for the fractional cut covering number of cubic graphs with sufficiently large girth.

2 New results

The main result of this paper is the following.

Theorem 1. *If G is a cubic graph with girth at least* $16\,353\,933$*, then there exists a probability distribution such that each edge of G is contained in an edge-cut drawn according to this distribution with probability at least* 0.88672*.*

Before we present the proof of Theorem 1, let us state three corollaries of this theorem. First, by considering the expected size of a cut drawn according to the distribution from Theorem 1, we get the following.

Corollary 2. *Every n-vertex cubic graph with girth at least* $16\,353\,933$ *contains an edge-cut of size at least* $1.33008n$*.*

Since a random cubic graph asymptotically almost surely contains only $o(n)$ cycles shorter than a fixed integer g [13], the lower bound on the size of an edge-cut also translates to random cubic graphs.

Corollary 3. *A random n-vertex cubic graph asymptotically almost surely contains an edge-cut of size at least* $1.33008n$*.*

Proof. Let G be a randomly chosen n-vertex cubic graph. The results of [13] imply then we can a.a.s. remove $o(n)$ vertices and obtain a subcubic subgraph G' with girth at least $16\,353\,933$. Let n_1 and n_2 be the numbers of vertices of G' with degree one and two, respectively. Observe that $n_1 + n_2 = o(n)$.

Let R be a $(2n_1 + n_2)$-regular graph with girth at least $16\,353\,933$. Replace each vertex of R with a copy of G' in such a way that the edges of R are incident with vertices of degree one and two in the copies of G' and the resulting graph is cubic. Observe that the obtained graph H has girth at least 16 353 933. Applying Corollary 2 to H yields an edge-cut C of size at least $1.33008N$ where N is the number of vertices of H. Observe that the number of the edges corresponding to those of R among all the edges of H is $o(N)$. Therefore, at least one copy of G' in H contains at least $1.33008n - o(n)$ edges of C. □

The last corollary relates Theorem 1 to the problem of fractional coverings the edges with edge-cuts. We show how to construct from the probability distribution given by Theorem 1 a fractional cut covering.

Corollary 4. *Every n-vertex cubic graph G with girth at least* 16 353 933 *has the fractional cut covering number at most* 1.12776.

Proof. Consider the probability distribution given by Theorem 1 for G. If the probability of a cut C to be drawn in this distribution is $p(C)$, assign C weight $p(C)/0.88672$. It is straightforward to verify that we have obtained a fractional cut covering of weight $1/0.88672 = 1.12776$. □

3 Structure of the proof

Our proof is inspired by the method which was developed by Hoppen in [5] for obtaining lower bounds on independent sets and induced forests. In order to prove Theorem 1, we design a randomized procedure for obtaining an edge-cut (of large size) which resembles the procedure used in [1]. The main difference between the procedures is that our procedure produces a cut where the parts of the cut have different sizes. The key tool for our analysis is the independence lemma (Lemma 6) which is given in Section 5. This lemma is used to simplify the recurrence relations appearing in the analysis. The recurrences describing the behavior of the randomized procedure are derived in Section 6. The actual performance of the procedure is based on setting up the parameters of the procedure and solving the recurrences numerically. This is discussed in Section 7.

The sought probability distribution is obtained by processing a cubic graph $G = (V, E)$ by the procedure which produces an edge-cut of it. G is processed in a fixed number of rounds K and the required assumption on the girth of G will depend only on the number K. We will iteratively construct two disjoint subsets $R \subseteq V$ and $B \subseteq V$; the vertices contained in R are referred to as red vertices and those in B as blue ones. The aim of the procedure is to maximize the number of red-blue edges. The vertices that are neither red nor blue will be called white.

All vertices are initially white. In every round, each white vertex is recolored to red or blue with a certain probability depending on the number of its red and blue neighbors, as well as on the number of current round. Once a vertex is colored red or blue, its color stays the same in all the remaining rounds of the procedure.

4 Detailed description

We now describe the randomized procedure in more detail. We first introduce some notation. Let $I_j := \{(r,b) : r \in \mathbb{N}_0, b \in \mathbb{N}_0, r+b \leq j\}$, i.e., the set I_j contains all pairs r and b of non-negative integers such that $r+b \leq j$. For example, $I_2 = \{(0,0),(0,1),(1,0),(1,1),(2,0),(0,2)\}$. Note that $|I_j| = \binom{j+2}{2}$. Let $G = (V,E)$ be a cubic graph and v a vertex of G. Throughout the analysis, $r(v)$ will refer to the number of red neighbors of v and $b(v)$ to the number of its blue neighbors. Therefore, $3 - r(v) - b(v)$ is the number of the white neighbors of v. If the vertex v is clear from the context, we just use r and b instead of $r(v)$ and $b(v)$.

Our randomized procedure is parametrized by the following parameters:

- an integer K,
- probabilities $P_k^{r,b}(W)$ for all $k \in [K]$ and $(r,b) \in I_3$,
- probabilities $P_k^{r,b}(R)$ for all $k \in [K]$ and $(r,b) \in I_3$ and
- probabilities $P_k^{r,b}(B)$ for all $k \in [K]$ and $(r,b) \in I_3$.

We require that $P_k^{r,b}(W) + P_k^{r,b}(R) + P_k^{r,b}(B) = 1$ for all $k \in [K]$ and $(r,b) \in I_3$.

The integer $K \in \mathbb{N}_0$ denotes the number of rounds that are performed. Throughout the procedure, vertices of the input graph G have one of the three colors: white (W), red (R) and blue (B). Let $W_k \subseteq V(G)$ denote the set of white vertices after the k-th round. Analogously, we define R_k and B_k as the sets of red vertices and blue vertices, respectively. As we have already mentioned, at the beginning of the process $W_0 := V, R_0 := \emptyset$ and $B_0 := \emptyset$. For $(r,b) \in I_3$ we define $W_k^{r,b} \subseteq W_k$ to be the set of white vertices with exactly r red neighbors and b blue neighbors. Hence the sets $W_k^{r,b}$ forms a partition of W_k for every k. Note that $W_0^{0,0} = V$ and $W_0^{r,b} = \emptyset$ for all $(r,b) \in I_3 \setminus \{(0,0)\}$.

Consider the coloring of G obtained after the k-th round. The $(k+1)$-th round of the procedure is performed as follows. Let v be a vertex from $W_k^{r,b}$.

With probability $P_{k+1}^{r,b}(R)$ we change the color of v to red, with probability $P_{k+1}^{r,b}(B)$ we recolor it to blue, and with probability $P_{k+1}^{r,b}(W)$ it remains white. If v is after the k-th round colored red or blue, it will not change its color during the $(k+1)$-th round.

Before we can proceed further, we have to introduce some additional notation. For a vertex $v \in V(G)$ let T_v^d denote the subgraph of G induced by vertices at the distance from v at most d. Observe that if the girth of G is larger than $2d+1$, then the subgraph T_v^d is a tree.

Now we show that if the girth of G is sufficiently large, then the probabilities that after the k-th round a vertex v has white, red or blue color, respectively, do not depend on the choice of v. We start with the following proposition.

Proposition 5. *Let G be a cubic graph with girth at least $2K$ and v a vertex of G. For every $k \in [K]$ the probability that the subgraph T_v^{K-k} has a certain coloring after the k-th round is determined by the coloring of T_v^{K-k+1} after the $(k-1)$-th round.*

Proof. The color of a vertex $u \in T_v^{K-k}$ after the k-th round depends only on the colors of u and its neighbors after the $(k-1)$-th round. Since all the neighbors of u are contained in T_v^{K-k+1}, the proposition follows. $\square$

Suppose that the girth of G is at least $2K$. For any $k \in [K]$ the structure of a subgraph T_v^{K-k} does not depend on the choice of v, i.e., it is always a tree with all inner vertices of degree three. Therefore, we use a simple inductive argument together with Proposition 5 to conclude that the following probabilities do not depend on the choice of v:

$$w_k := \mathbf{P}\big[v \in W_k\big] \ , \quad r_k := \mathbf{P}\big[v \in R_k\big] \ , \quad b_k := \mathbf{P}\big[v \in B_k\big] \ .$$

Analogously, for any $k \in [K-1]$ and $(r,b) \in I_3$, the probability that after the k-th round a vertex v is white and has r red neighbors and b blue neighbors does not depend on the choice of v as well. Therefore, we can define

$$w_k^{r,b} := \mathbf{P}\big[v \in W_k^{r,b} \bigm| v \in W_k\big] \ .$$

Also observe that if the girth of G is larger than $2d+2$, then for every edge $uv \in E(G)$ the subgraph of G induced by vertices x satisfying $\min\{d(x,u), d(x,v)\} \leq d$ is a tree. If the girth of G is at least $2K+1$, the same reasoning as before yields the following. The probability that for an edge $uv \in E(G)$ either u is red and v is blue after the k-th round, or v is red and u is blue after the k-th round does not depend on the choice of uv. This probability will be denoted by

$$p_k := \mathbf{P}\left[(u \in R_k \wedge v \in B_k) \vee (u \in B_k \wedge v \in R_k) \;\middle|\; uv \in E(G)\right] .$$

5 Independence lemma

In this section we present a key tool which we use in the analysis of the randomized procedure. Our analysis follows the approach used in [5].

We start with a definition. If G is a cubic graph with girth at least $2K+1$, uv is an edge of G and d is an integer between 0 and $K-1$, $T^d_{v,u}$ denotes the component of $T^d_v - u$ containing the vertex v. We refer to v as to the root of $T^d_{v,u}$. From the assumption on the girth it follows that all the subgraphs $T^d_{v,u}$ are isomorphic to the same rooted binary tree $\mathcal{T}^d$ of depth d.

Let $k \in [K]$. For a set $V' \subseteq V(G)$ let $c_k(V')$ denote the coloring of vertices V' after the k-th round. The set of all colorings of $\mathcal{T}^{K-k}$ such that the root of the tree is white is denoted by $\mathcal{C}_k$. Observe that by the girth assumption for any $\gamma \in \mathcal{C}_k$ the probability $\mathbf{P}\left[c_k\left(T^{K-k}_{v,u}\right) = \gamma\right]$ does not depend on the edge uv.

We are ready to prove the main lemma of this section.

Lemma 6 (Independence lemma)**.** *Consider the randomized procedure with parameters K and $P^{r,b}_i(C)$, where $i \in [K], (r,b) \in I_3$ and $C \in \{W, R, B\}$. Let G be a cubic graph with girth at least $2K+1$, uv an edge of G, k an integer smaller than K and γ_u and γ_v two colorings from $\mathcal{C}_k$. Conditioned by the event $uv \in W_k$, the events $c_k\left(T^{K-k}_{v,u}\right) = \gamma_v$ and $c_k\left(T^{K-k}_{v,u}\right) = \gamma_u$ are independent. In other words, the probabilities*

$$\mathbf{P}\left[c_k\left(T^{K-k}_{v,u}\right) = \gamma_v \;\middle|\; uv \subseteq W_k\right] \tag{1}$$

and

$$\mathbf{P}\left[c_k\left(T^{K-k}_{v,u}\right) = \gamma_v \;\middle|\; v \in W_k \wedge c_k\left(T^{K-k}_{u,v}\right) = \gamma_u\right] \tag{2}$$

are equal.

Proof. The proof proceeds by induction on k. After the first round each vertex has a color C with probability $P^{0,0}_1(C)$ independently of the colors of the other vertices. Hence, the claim holds for $k = 1$.

Assume now that $k > 1$. By the definition of the conditional probability and the fact that the event $uv \subseteq W_k$ immediately implies that the event $uv \subseteq W_{k-1}$ occurs, (1) is equal to

$$\frac{\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \wedge u \in W_k \;\middle|\; uv \subseteq W_{k-1}\right]}{\mathbf{P}\left[uv \subseteq W_k \;\middle|\; uv \subseteq W_{k-1}\right]} . \tag{3}$$

Analogously, (2) is equal to

$$\frac{\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \wedge c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \;\middle|\; uv \subseteq W_{k-1}\right]}{\mathbf{P}\left[v \in W_k \wedge c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \;\middle|\; uv \subseteq W_{k-1}\right]} . \tag{4}$$

Now we expand the numerator of (3).

$$\begin{aligned}
&\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\left[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \;\middle|\; uv \subseteq W_{k-1}\right] \\
&\times \mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \;\middle|\; v \in W_{k-1} \wedge c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u\right] \\
&\times \mathbf{P}\left[u \in W_k \;\middle|\; c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v\right] \\
&\times \mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \;\middle|\; c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_k\right] .
\end{aligned}$$

By the induction hypothesis, for any two colorings $\gamma'_u, \gamma'_v \in \mathcal{C}_{k-1}$ the probabilities

$$\mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \;\middle|\; v \in W_{k-1} \wedge c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u\right]$$

and

$$\mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \;\middle|\; uv \subseteq W_{k-1}\right]$$

are equal.

Since the new color of u is determined only by the colors of the neighbors of u, it follows that the probabilities

$$\mathbf{P}\left[u \in W_k \;\middle|\; c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v\right]$$

and

$$\mathbf{P}\left[u \in W_k \;\middle|\; c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\right]$$

are also equal.

Analogously, for any vertex $w \in T_{v,u}^{K-k} \setminus \{v\}$ the new color of w does not depend on γ'_u at all. Applying the same reasoning for v yields that the probabilities

$$\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \;\middle|\; c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_k\right]$$

and

$$\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right)=\gamma_v \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v \wedge u\in W_{k-1}\big]$$

are equal as well. Note that in the last equality we have also used that the random choices of new colors for two arbitrary vertices in the $(k+1)$-th round are independent.

By changing the order of summation, we conclude that the numerator of (3) is equal to

$$\begin{aligned}&\Bigg(\sum_{\gamma'_u\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right)=\gamma'_u \mid uv\subseteq W_{k-1}\big]\\&\qquad\times\mathbf{P}\big[u\in W_k \mid c_{k-1}\left(T_{u,v}^{K-k+1}\right)=\gamma'_u\wedge v\in W_{k-1}\big]\Bigg)\\&\times\Bigg(\sum_{\gamma'_v\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\\&\qquad\times\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right)=\gamma_v \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v\wedge u\in W_{k-1}\big]\Bigg).\end{aligned}$$

Along the same lines, the denominator of (3) is equal to

$$\begin{aligned}&\Bigg(\sum_{\gamma'_u\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right)=\gamma'_u \mid uv\subseteq W_{k-1}\big]\\&\qquad\times\mathbf{P}\big[u\in W_k \mid c_{k-1}\left(T_{u,v}^{K-k+1}\right)=\gamma'_u\wedge v\in W_{k-1}\big]\Bigg)\\&\times\Bigg(\sum_{\gamma'_v\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\\&\qquad\times\mathbf{P}\big[v\in W_k \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v\wedge u\in W_{k-1}\big]\Bigg).\end{aligned}$$

Canceling out the sum over γ'_u which is the same in both numerator and denominator of (3), we derive that (1) is equal to

$$\frac{\sum\limits_{\gamma'_v\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\times\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right)=\gamma_v \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v\wedge u\in W_{k-1}\big]}{\sum\limits_{\gamma'_v\in\mathcal{C}_{k-1}}\mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\times\mathbf{P}\big[v\in W_k \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right)=\gamma'_v\wedge u\in W_{k-1}\big]}. \tag{5}$$

We apply the same trimming to the numerator and denominator of (4). The numerator is first expanded to

$$\left(\sum_{\gamma'_u\in\mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T^{K-k+1}_{u,v}\right)=\gamma'_u \mid uv\subseteq W_{k-1}\big]\right.$$
$$\left.\times\,\mathbf{P}\big[c_k\left(T^{K-k}_{u,v}\right)=\gamma_u \mid c_{k-1}\left(T^{K-k+1}_{u,v}\right)=\gamma'_u\wedge v\in W_{k-1}\big]\right)$$
$$\times\left(\sum_{\gamma'_v\in\mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T^{K-k+1}_{v,u}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\right.$$
$$\left.\times\,\mathbf{P}\big[c_k\left(T^{K-k}_{v,u}\right)=\gamma_v \mid c_{k-1}\left(T^{K-k+1}_{v,u}\right)=\gamma'_v\wedge u\in W_{k-1}\big]\right)$$

and the denominator is then expanded to

$$\left(\sum_{\gamma'_u\in\mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T^{K-k+1}_{u,v}\right)=\gamma'_u \mid uv\subseteq W_{k-1}\big]\right.$$
$$\left.\times\,\mathbf{P}\big[c_k\left(T^{K-k}_{u,v}\right)=\gamma_u \mid c_{k-1}\left(T^{K-k+1}_{u,v}\right)=\gamma'_u\wedge v\in W_{k-1}\big]\right)$$
$$\times\left(\sum_{\gamma'_v\in\mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T^{K-k+1}_{v,u}\right)=\gamma'_v \mid uv\subseteq W_{k-1}\big]\right.$$
$$\left.\times\,\mathbf{P}\big[v\in W_k \mid c_{k-1}\left(T^{K-k+1}_{v,u}\right)=\gamma'_v\wedge u\in W_{k-1}\big]\right).$$

By canceling out the sum over γ'_u, we obtain (5). Therefore the expressions (1) and (2) are equal. □

6 Recurrence relations

In this section we derive recurrence relations for the probabilities describing the behavior of the randomized procedure.

Fix parameters K and $P^{r,b}_k(C)$, $k\in[K],(r,b)\in I_3$ and $C\in\{W,R,B\}$. We will inductively show that the probabilities describing the state of the procedure after the $(k+1)$-th round can be computed using only the probabilities describing the state after the k-th round. This yields the recurrence relations for the probabilities, which is the main goal of this section.

We start with determining the probabilities after the initialization round. It is easy to see that the probabilities r_1, b_1, w_1, p_1 and $w_1^{r,b}$ are

$$
\begin{aligned}
&r_1 = P_1^{0,0}(R) \ , \\
&b_1 = P_1^{0,0}(B) \ , \\
&w_1 = 1 - r_1 - b_1 \ , \\
&p_1 = 2 \cdot P_1^{0,0}(R) \cdot P_1^{0,0}(B) \quad \text{and} \\
&w_1^{r,k} = \binom{3}{r}\binom{3-r}{b} \cdot \Big(P_1^{0,0}(R)\Big)^r \cdot \Big(P_1^{0,0}(B)\Big)^b \cdot \Big(1 - P_1^{0,0}(R) - P_1^{0,0}(B)\Big)^{3-r-b}
\end{aligned}
$$

for $(r, b) \in I_3$.

Now we show how to compute the probabilities r_{k+1}, b_{k+1} and w_{k+1} from r_k, b_k, w_k and $w_k^{r,b}$. We start with the formula for r_{k+1}. If a vertex v is colored red after the $(k+1)$-th round, then after the k-th round, it was either already colored red, or it was white, had r red neighbors, b blue neighbors and it was recolored to red. The latter happened with probability $P_{k+1}^{r,b}(R)$. The probability of the first event is r_k and that of the second event is $w_k \cdot w_k^{r,b} \cdot P_{k+1}^{r,b}(R)$. This yields that

$$r_{k+1} = r_k + w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(R) \ .$$

Analogously, we can compute

$$b_{k+1} = b_k + w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(B) \ ,$$

and finally w_{k+1} is given by

$$w_{k+1} = 1 - r_{k+1} - b_{k+1} \ .$$

Before we proceed with the recurrences for p_{k+1} and $w_{k+1}^{r,b}$, let us introduce some auxiliary notation. All of the following quantities are fully determined by $w_k^{r,b}$, but this notation will help to make the formulas simpler. We start with probability that a vertex v has white color after the $(k+1)$-th round conditioned by the event it had white color after the k-th round. This quantity will be denoted by $w_{\to k+1}$. It is straightforward to check that

$$w_{\to k+1} := \mathbf{P}\big[v \in W_{k+1} \bigm| v \in W_k\big] = \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(W) \ .$$

Next, we consider the probability that the vertex u is white after the k-th round conditioned by the event that a fixed neighbor v of u is white. This will be denoted by q_k^{W-W}. Again it is easy to check that

$$q_k^{W-W} := \mathbf{P}\big[uv \subseteq W_k \bigm| v \in W_k\big] = \sum_{(r,b)\in I_2} \frac{3-r-b}{3} \cdot w_k^{r,b} \ .$$

Finally, for a color $C \in \{W, R, B\}$ and an edge $e = uv$, $q_{\to k+1}^{(C)}$ denotes the probability that u has the color C after the $(k+1)$-th round conditioned by the event that both u and v were white after the k-th round. We infer from the definition of the conditional probability that

$$q_{\to k+1}^{(R)} := \mathbf{P}\big[u \in R_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(R)}{3 \cdot q_k^{W-W}} \ ,$$

$$q_{\to k+1}^{(B)} := \mathbf{P}\big[u \in B_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(B)}{3 \cdot q_k^{W-W}} \ ,$$

$$q_{\to k+1}^{(W)} := \mathbf{P}\big[u \in W_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(W)}{3 \cdot q_k^{W-W}} \ .$$

Now we are ready to present the remaining recurrences. We start with p_{k+1}, i.e., the probability than an edge $e = uv$ is red-blue after the $(k+1)$-th round. Note that once we color a vertex x with either red or blue color, the color of x in the future rounds will stay the same. Therefore, we can split the contribution to p_{k+1} to the following four types.

1. $e \cap W_k = \emptyset$: This event happens with probability p_k and the colors stay the same.

2. $e \cap W_k = \{v\}$: Suppose first that u is blue. The probability that we have such configuration after k-th round is $w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot b/3$. In this case, the edge e become red-blue after the $(k+1)$-th round with probability $P_{k+1}^{r,b}(R)$. Analogously, if u is red, the contribution of this case is $w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(B) \cdot r/3$.

3. $e \cap W_k = \{u\}$: This case is symmetric to the previous one.

4. $e \subseteq W_k$: The probability that v has white color is w_k. With probability $w_k^{r,b} \cdot (3-r-b)/3$, v has r red neighbors, b blue neighbors and u is white. The probability that v becomes red is $P_{k+1}^{r,b}(R)$, and using

the independence lemma (Lemma 6) the neighborhood of u does not depend on the colors of the other neighbors of v. Therefore, the probability that u becomes blue is $q^{(B)}_{\to k+1}$. On the other hand, the probability that v becomes red and u becomes blue is $P^{r,b}_{k+1}(B) \cdot q^{(R)}_{\to k+1}$.

The analysis just presented yields that

$$p_{k+1} = p_k + \frac{w_k}{3} \cdot \sum_{(r,b)\in I_3} w^{r,b}_k \cdot P^{r,b}_{k+1}(R) \cdot \Big(2b + (3-r-b) \cdot q^{(B)}_{\to k+1}\Big) + \frac{w_k}{3} \cdot \sum_{(r,b)\in I_3} w^{r,b}_k \cdot P^{r,b}_{k+1}(B) \cdot \Big(2r + (3-r-b) \cdot q^{(R)}_{\to k+1}\Big) .$$

We finish this section with the recurrence relations for the probabilities $w^{r,b}_{k+1}$. Observe that

$$w^{r,b}_{k+1} = \frac{\mathbf{P}\big[v \in W^{r,b}_{k+1}\big]}{\mathbf{P}\big[v \in W_{k+1}\big]} = \frac{\mathbf{P}\big[v \in W^{r,b}_{k+1} \mid v \in W_k\big]}{\mathbf{P}\big[v \in W_{k+1} \mid v \in W_k\big]} . \tag{6}$$

The second equality holds because each of the events $v \in W_{k+1}$ and $v \in W^{r,b}_{k+1}$ immediately implies that the event $v \in W_k$ occurs. The denominator of (6) is equal to $w_{\to k+1}$, so it remains to derive the formula for the numerator.

Let $N^W_k(v)$ denote the set of white neighbors of v after the k-th round. Using the same argument as for deriving the formula for p_{k+1}, the color after the $(k+1)$-th round of a white neighbor $u \in N^W_k(v)$ will be red with probability $q^{(R)}_{\to k+1}$. Analogously, it will be blue with probability $q^{(B)}_{\to k+1}$ and white with probability $q^{(W)}_{\to k+1}$. Finally, by Lemma 6 and the fact that in all rounds we recolor each white vertex independently of the others, the new color of a neighbor $u_1 \in N^W_k(v)$ does not depend on the new color of another neighbor $u_2 \in N^W_k(v)$. Therefore, it holds for each $(r,b) \in I_3$ that

$$w^{r,b}_{k+1} = \sum_{\substack{\overline{r}\le r\\ \overline{b}\le b}} \frac{w^{\overline{r},\overline{b}}_k \cdot P^{\overline{r},\overline{b}}_{k+1}(W) \cdot \binom{3-\overline{r}-\overline{b}}{r-\overline{r}}\binom{3-r-\overline{b}}{b-\overline{b}} \cdot \left(q^{(R)}_{\to k+1}\right)^{r-\overline{r}} \cdot \left(q^{(B)}_{\to k+1}\right)^{b-\overline{b}} \cdot \left(q^{(W)}_{\to k+1}\right)^{3-r-b}}{w_{\to k+1}} ..$$

7 Setting up the parameters

In this section we set up the parameters in the randomized procedure. In the first round, we pick a vertex with a small probability p_0 and color it either red or blue. The next rounds of the procedure are split into two phases, which consist of K_1 and K_2 rounds, respectively. Therefore, the total number of rounds K is equal to $K_1 + K_2 + 1$.

In the rounds of the first phase, with probability p_B (p_R), where $p_R \ll p_B$, we color a vertex with exactly one red (blue) neighbor by blue (red). If a vertex has at least two neighbors of the same color, we color it with the other color with probability one. In all the other cases we do nothing.

With one exception, the rounds of the second phase are performed identically to the rounds of the first phase. The exception is that a white vertex with one red, one blue and one white neighbor is colored red with probability $p_{RB}/2$ or blue with probability $p_{RB}/2$. The choice of p_{RB} is such that $p_{RB} \ll p_R$.

Specifically, we set:

- $K := K_1 + K_2 + 1$,
- $P_1^{0,0}(R) := p_0/2$, $P_1^{0,0}(B) := p_0/2$,
- $P_k^{r,b}(R) := 1$ for $(r,b) \in I_3 \cap \{(r,b) : b \geq 2\}$ for $k \in [2,\ldots,K]$,
- $P_k^{r,b}(W) := 1$ for $(r,b) \in I_3 \cap \{(r,b) : r \geq 2\}$ for $k \in [2,\ldots,K]$,
- $P_k^{0,1}(R) := p_R$, $P_k^{1,0}(B) := p_B$ for $k \in [2,\ldots,K]$,
- $P_k^{1,1}(R) := p_{RB}/2$, $P_k^{1,1}(B) := p_{RB}/2$ for $k \in [K_1+2,\ldots,K]$,
- $P_k^{r,b}(R) := 0$ for all the other choices of r and b,
- $P_k^{r,b}(B) := 0$ for all the other choices of r and b and
- $P_k^{r,b}(W) := 1 - P_k^{r,b}(R) - P_k^{r,b}(B)$ for $(r,b) \in I_3$.

The recurrences presented in this chapter were solved numerically using a computer program. The particular choice of parameters used in the program was $p_0 = 10^{-5}, p_B = 1/10, p_R = 10^{-5}, p_{RB} = 10^{-6}$, $K_1 = 1\,672\,413$ and $K_2 = 6\,504\,552$.

The choice of K_1 was made in such a way that at the end of the first phase, i.e. after the (K_1+1)-th round, the probability that a vertex is white and has exactly one non-white neighbor is less than 10^{-7}. Analogously, the choice of K_2 was made in a way that at the end of the process, i.e. after the K-th round, the probability that a vertex is white is less than 10^{-7}. We also estimated the precision of our calculations based on the representation of float numbers to avoid rounding errors affecting the presented bound on significant digits. Solving the recurrences for the above choice of parameters we have obtained that $p_K > 0.88672$.

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