Extensions of Fractional Precolorings show Discontinuous Behavior *

Jan van den Heuvel † Daniel Král' ‡ Martin Kupec ‡
Jean-Sébastien Sereni § Jan Volec ‡

Abstract

We study the following problem: given a real number k and integer d, what is the smallest ε such that any fractional $(k+\varepsilon)$-precoloring of vertices at pairwise distances at least d of a fractionally k-colorable graph can be extended to a fractional $(k+\varepsilon)$-coloring of the whole graph? The exact values of ε were known for $k \in \{2\} \cup [3, \infty)$ and any d. We determine the exact values of ε for $k \in (2, 3)$ if $d = 4$, and $k \in [2.5, 3)$ if $d = 6$, and give upper bounds for $k \in (2, 3)$ if $d = 5, 7$, and $k \in (2, 2.5)$ if $d = 6$. Surprisingly, ε viewed as a function of k is discontinuous for all those values of d.

1 Introduction and main results

Graph coloring is one of the classical topics in graph theory. In this paper, we seek conditions when a precoloring of some vertices in a graph can be extended to a coloring of the entire graph. This line of research was initiated by Thomassen [17] who asked for sufficient conditions on extending precolorings of vertices in planar graphs. His original question led to the following result of Albertson [1].

Theorem 1.1 ([1]). *Let G be an r-colorable graph and W a subset of its vertex set such that the distance between any two vertices of W is at least four. Then every $(r+1)$-coloring of W can be extended to an $(r+1)$-coloring of G.*

* Research for this paper was started during visits of JvdH, DK, MK and JV to LIAFA. The authors like to thank the members of LIAFA for their hospitality. Visits of MK and JV to LIAFA and visits of J-SS to Charles University were supported by the PHC Barrande 24444 XD.

† Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, UK. Email: jan@maths.lse.ac.uk.

‡ Computer Science Institute, Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 118 00 Prague 1, Czech Republic. DK was supported by the project P202/12/G061 CE-ITI.. MK was supported by the grant GAUK 60310. E-mail: {kral,kupec,volec}@iuuk.mff.cuni.cz.

§ CNRS (LIAFA, Université Denis Diderot), Paris, France. E-mail: sereni@kam.mff.cuni.cz.

This result initiated a line of research [2, 3, 4, 5, 6, 8] seeking conditions for the existence of an extension of a precoloring of various types of subgraphs.

It is natural to ask whether an analogue of Theorem 1.1 also holds for non-integer relaxations of colorings. For circular colorings introduced in [18], the extension problem was almost completely solved by Albertson and West [7] (see [19, 20] for background and results on circular colorings).

Another well-established relaxation of classical colorings is the notion of fractional colorings, see [15], which we address in this paper. A *fractional k-coloring* of a graph G is an assignment of measurable subsets of the interval $[0, k) \subseteq \mathbb{R}$ to the vertices of G such that each vertex receives a subset of measure one and adjacent vertices receive disjoint subsets. The *fractional chromatic number of* G is the infimum over all positive real numbers k such that G admits a fractional k-coloring. For finite graphs (which we restrict our attention to), such k exists, the infimum is in fact a minimum, and its value is always rational. A *fractional k-precoloring* is an assignment of measurable subsets of measure one of the interval $[0, k)$ to some vertices of a graph.

In this paper, we study conditions under which a fractional precoloring can be completed to a fractional coloring of the whole graph.

Problem 1. *Let $\varepsilon > 0$ be a real, $k \geq 2$ a rational and $d \geq 3$ an integer. Given a fractionally k-colorable graph G and a fractional $(k+\varepsilon)$-precoloring of a subset of its vertex set at pairwise distance at least d, is it possible to extend the precoloring to a fractional $(k + \varepsilon)$-coloring of the whole graph G?*

For a fixed rational $k \geq 2$ and a fixed integer $d \geq 3$, let $g(k, d)$ be the infimum over all non-negative reals satisfying the following: for any $\varepsilon \geq g(k, d)$ and any fractionally k-colorable graph G, an arbitrary $(k+\varepsilon)$-precoloring of vertices at pairwise distance at least d in G can be extended to a fractional $(k + \varepsilon)$-coloring of G. The next proposition, which is proved in [12], implies that for any $\varepsilon < g(k, d)$ there exists a fractionally k-colorable graph G with a fractional $(k + \varepsilon)$-precoloring of some of its vertices at pairwise distance at least d, such that there is no extension of the precoloring to a fractional $(k + \varepsilon)$-coloring of G.

Proposition 1.2 ([12])**.** *Let G be a graph with fractional chromatic number k and W a subset of its vertex set. The set of all non-negative reals ε such that any fractional $(k + \varepsilon)$-precoloring of W can be extended to a fractional $(k + \varepsilon)$-coloring of G is a closed interval.*

The only value of d for that the values of $g(k, d)$ are known for all $k \geq 2$ is $d = 3$. In this case, $g(k, 3) = 1$ for all $k \in [2, \infty)$, see [12]. For $d \geq 4$, the values of $g(k, d)$ for $k \in \{2\} \cup [3, \infty)$ were determined in [12].

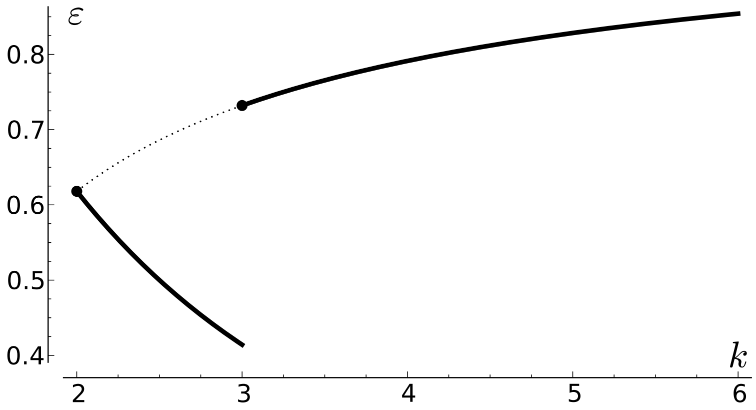

Figure 1: The values of $g(k,4)$.

Theorem 1.3 ([12])**.** *For every $k \in \{2\} \cup [3,\infty)$ and $d \geq 3$, we have:*

$$g(k,d) = \begin{cases} \dfrac{\sqrt{(d'k-1)^2 + 4d'(k-1)} - (d'k-1)}{2d'}, & \text{if } d \equiv 0 \bmod 4; \\ \dfrac{k-1}{d'k}, & \text{if } d \equiv 1 \bmod 4; \\ \dfrac{\sqrt{(d'k)^2 + 4d'(k-1)} - d'k}{2d'}, & \text{if } d \equiv 2 \bmod 4; \\ \dfrac{k-1}{d'k+k-1}, & \text{otherwise,} \end{cases}$$

where $d' = \lfloor d/4 \rfloor$. The formula also holds for $k \in [2,\infty)$ and $d = 3$.

The main goal of this paper is to put more light on values of $g(k,d)$ for $k \in (2,3)$. We determine the values of $g(k,d)$ for $k \in (2,3)$ if $d = 4$, and for $k \in [2.5,3)$ if $d = 6$ (see Figures 1 and 3).

Theorem 1.4. *For $k \in [2,3)$ we have $g(k,4) = \frac{1}{2}\big(\sqrt{(k-1)^2+4} - k + 1\big)$.*

Theorem 1.5. *For $k \in \{2\} \cup [2.5,3)$ we have $g(k,6) = \frac{1}{2}\big(\sqrt{k^2+4} - k\big)$.*

For additional values of $k \in (2,3)$ and d, we provide upper bounds (Theorems 3.2, 4.2, 5.2, 6.2, and 6.3) which we believe to be tight. See Figures 2 and 4 for the bounds we can prove for $d = 5$ and $d = 7$. To our surprise, for fixed $d \in \{4,5,6,7\}$, the function $g(k,d)$ is discontinuous in k at $k = 3$, while for $d \in \{6,7\}$ the function $g(k,d)$ is also discontinuous at $k = 2.5$.

The paper is organized as follows. In the analysis of the values of $g(k,d)$, we consider four cases based on the remainder of d modulo 4. In Section 3, we present our upper bounds on $g(k,d)$ for $k \in (2,3)$ and d divisible by four. We also present the matching lower bound

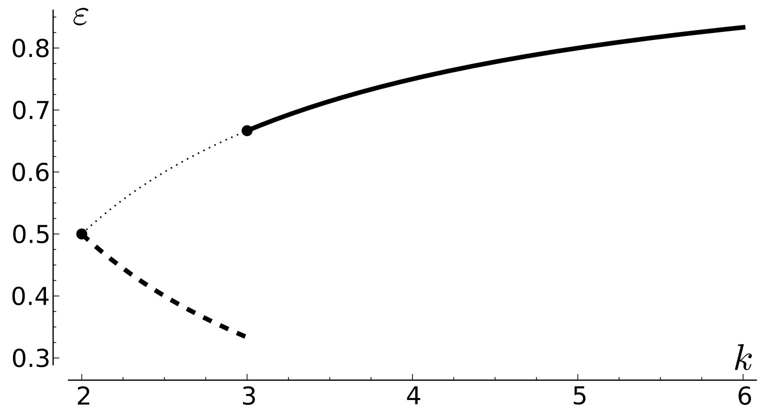

Figure 2: The values of $g(k, 5)$. Upper bounds are given by dashed lines.

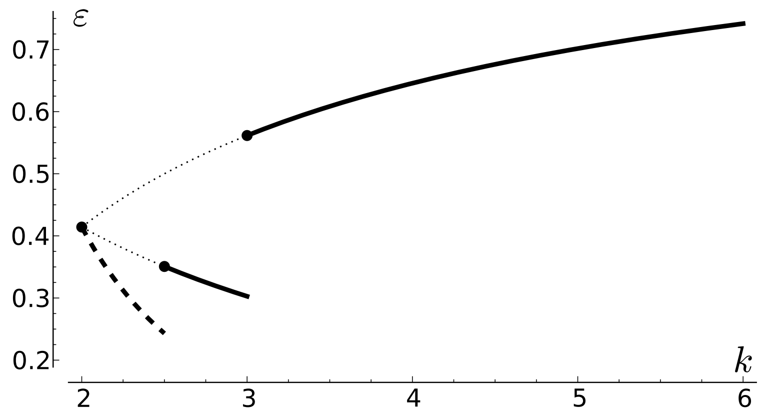

Figure 3: The values of $g(k, 6)$. Upper bounds are given by dashed lines.

for $d = 4$. This lower bound is based on a simple expansion bound on independent sets in Kneser graphs based on eigenvalues of its adjacency matrix. In Section 4, we present our upper bounds on $g(k, d)$ for $k \in (2, 3)$ and d congruent to two modulo four. This section also contains the matching lower bound for the case $d = 6$ and $k \in [2.5, 3)$. This lower bound uses a suitable solution of the linear program dual to that for finding the fractional chromatic number of a Kneser graph. Finally, in Sections 5 and 6 we present our upper bounds on $g(k, d)$ for d congruent to one and three, respectively.

2 Notation, definitions and preliminary results

Before we can present our results, and their proofs, in detail, we need to introduce some notation. For a positive integer n, we set $[n] := \{1, \dots, n\}$. Next, for a set $Y \subseteq [0, \infty)$ we

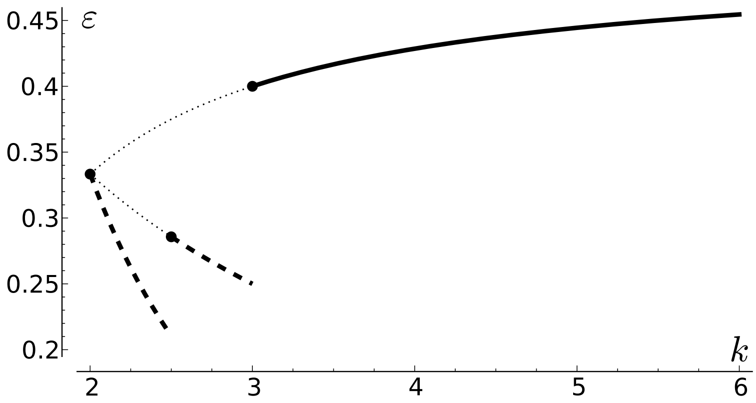

Figure 4: The values of $g(k, 7)$. Upper bounds are given by dashed lines.

write 2^Y for the set of all measurable subsets of Y. If $f : X \to 2^Y$ is a mapping from a set X to 2^Y and A is a subset of X, we write $f(A)$ for the set $\bigcup_{a\in A} f(a)$. We also write $g : X \hookrightarrow 2^Y$ for mappings from X to 2^Y such that $g(i) \cap g(j) = \varnothing$ for any two distinct $i, j \in X$.

We gave one possible definition of the fractional chromatic number of a graph G in the introduction. An equivalent definition can be given as a linear relaxation of determining the ordinary chromatic number: assign non-negative real weights to the independent sets of G such that for every vertex $v \in V(G)$ the sum of the weights of independent sets containing v is at least one. The minimum possible sum of weights of all independent sets in G, where the minimum is taken over all such assignments, is equal to the fractional chromatic number of G.

The definition of fractional colorings also allows one to define a class of *universal* graphs, i.e., a class such that for every graph with fractional chromatic number k there is a homomorphism to one of the graphs in this class. A *homomorphism* from a graph G to a graph H is a mapping $f : V(G) \to V(H)$ such that if u and v are two adjacent vertices of G, then the vertices $f(u)$ and $f(v)$ are adjacent in H. If such a mapping exists, we say that G is *homomorphic* to H.

Universal graphs for fractional colorings are Kneser graphs $K_{p/q}$; the graph $K_{p/q}$, for integers $1 \le q \le p$, has a vertex set formed by all q-element subsets of $[p]$, i.e., $V(K_{p/q}) = \binom{[p]}{q}$. Two vertices A and A' are adjacent if $A \cap A' = \varnothing$. It is not hard to show that the fractional chromatic number of $K_{p/q}$ is equal to p/q. The following proposition can be found, e.g., in [9].

Proposition 2.1. *Let G be a graph with fractional chromatic number k. There exist integers p and q such that $k = p/q$ and G is homomorphic to the graph $K_{p/q}$.*

Analogously to [12], our proofs are based on defining and analyzing graphs that are

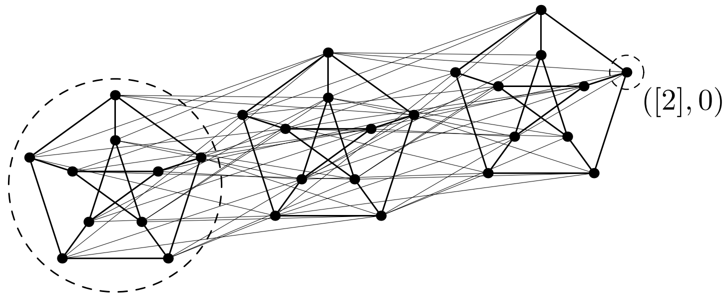

Figure 5: The ray $R^{[2]}_{5,2,2}$.

universal for graphs (of a given fractional chromatic number) with some precolored vertices. The graphs we introduce now are isomorphic to the ones defined in [12], although we use a slightly different notation.

The *extension product* of graphs G and H is the graph with vertex set $V(G)\times V(H)$ such that vertices (u,v) and (u',v') are adjacent if u and u' are adjacent in G and either $v=v'$, or v and v' are adjacent in H. This type of a graph product was introduced by Albertson and West [7]. An equivalent notion was used in [12] under the name *universal product*; the only difference is that the meaning of G and H was swapped, i.e., the universal product of G and H is isomorphic to the extension product of H and G. For a set $X \in \binom{[p]}{q}$, a *ray* $R^X_{p,q,P}$ is the extension product of the Kneser graph $K_{p/q}$ and the $(P+1)$-vertex path with vertices $0,\dots,P$; the vertex $(X,0)$ of $R^X_{p,q,P}$ is marked as *special*. The copy of $K_{p/q}$ in the ray $R^X_{p,q,P}$ corresponding to the vertex P of the path is said to be the *base* of the ray. For brevity, $R_{p,q,P}$ will stand for $R^{[q]}_{p,q,P}$ in what follows. The ray $R^{[2]}_{5,2,2}$ is sketched in Figure 5. Note that the graph $R^X_{p,q,P}$ is homomorphic to $K_{p/q}$, and the distance of the vertex $(X,0)$ from any vertex (A,ℓ), for $A \in \binom{[p]}{q}$ and $\ell \in [1,P]$, is at least ℓ.

The graph $U^n_{p,q,d}$, which we now define, is a universal graph for graphs with fractional chromatic number p/q with n precolored vertices at pairwise distance at least d. Fix positive integers p,q,d and n such that $p \geq q/2$ and $d \geq 3$. If d is even, the graph $U^n_{p,q,d}$ is the extension product of the Kneser graph $K_{p/q}$ and the star $K_{1,n\binom{[p]}{q}}$ with each edge subdivided $d/2-1$ times. For every $X \in \binom{[p]}{q}$, we mark the vertex X as *special* in n copies of $K_{p/q}$ corresponding to the leaves of the star (for different values of X, we choose different copies). In this way, the subgraphs of $U^n_{p,q,d}$ corresponding to the products of the subdivided paths and $K_{p/q}$ are isomorphic to rays $R^X_{p,q,d/2}$. Hence, the graph $U^n_{p,q,d}$ can be viewed as obtained from n copies of the ray $R^X_{p,q,d/2}$ for each choice of $X \in \binom{[p]}{q}$ through identification of the bases of the rays. The graph $U^1_{5,2,6}$ is sketched in Figure 6.

For positive integers N and P, let $L_{N,P}$ be the graph obtained from a clique K_N by

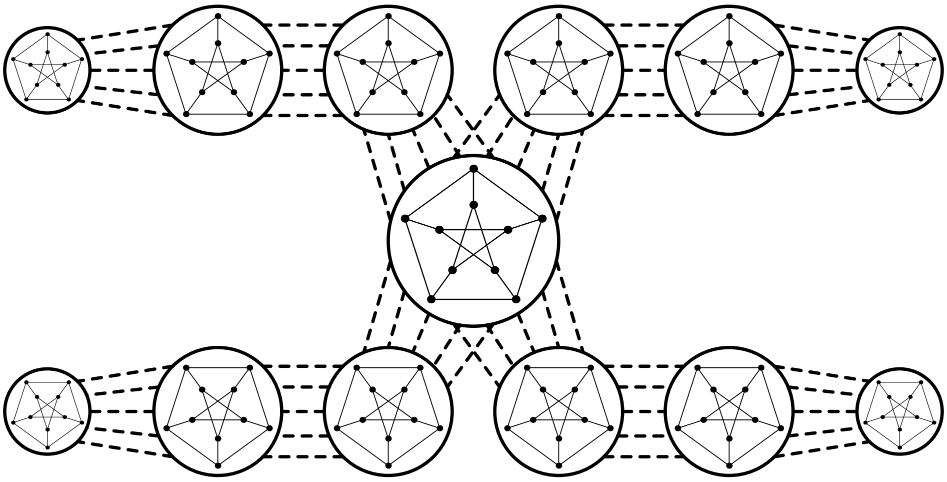

Figure 6: A sketch of the graph $U^1_{5,2,6}$ (some rays are omitted).

identifying each vertex of the clique with an end-vertex of a P-vertex path; so $L_{N,P}$, for $P \geq 2$, has $N \cdot (P-2)$ vertices of degree two, N vertices of degree one, and N vertices of degree N. If d is odd, the graph $U^n_{p,q,d}$ is the extension product of the Kneser graphs $K_{p/q}$ and the graph $L_{n\binom{p}{q},(d+1)/2}$. Again, for each $X \in \binom{[p]}{q}$, we mark vertices X in n of the copies of $K_{p/q}$ corresponding to the vertices of degree one of $L_{n\binom{p}{q},(d+1)/2}$ as special (with different copies for different values of X again). In this way, we can view $U^n_{p,q,d}$ as a union of $n\binom{p}{q}$ rays $R^X_{p,q,(d-1)/2}$ with additional edges between their bases. The graph $U^1_{5,2,7}$ is sketched in Figure 7.

In the next three propositions, we summarize the properties of the graphs $U^n_{p,q,d}$ needed in the proofs. We start with the first two of them; the proof of the first one is straightforward and the proof of the second one is in [12].

Proposition 2.2. *The graph $U^n_{p,q,d}$ for $p/q \geq 2$ and $d \geq 3$ is homomorphic to $K_{p/q}$ and its special vertices are at pairwise distance at least d.*

Proposition 2.3 ([12])**.** *Let G be a graph with fractional chromatic number k and W a subset of its vertex set at pairwise distance at least $d \geq 3$. There exist positive integers p and q, such that $k = p/q$ and the graph G has a homomorphism to $U^{|W|}_{p,q,d}$ that maps the vertices of W to special vertices of $U^{|W|}_{p,q,d}$.*

The length of the shortest odd cycle of a graph G is the *odd girth* of G. The odd girth of the Kneser graph $K_{p/q}$ is equal to $2\left\lceil \frac{q}{p-2q} \right\rceil + 1$, see [13]. Note that Proposition 2.1 implies that if G is a fractionally k-colorable graph, then its odd girth is at least $2\lceil 1/(k-2) \rceil + 1$. The main difference between the case $k \in \{2\} \cup [3, \infty)$, which was fully analyzed in [12], and the case $k \in (2, 3)$ is that vertices of a ray $R_{p,q,P}$ at some fixed small distance from

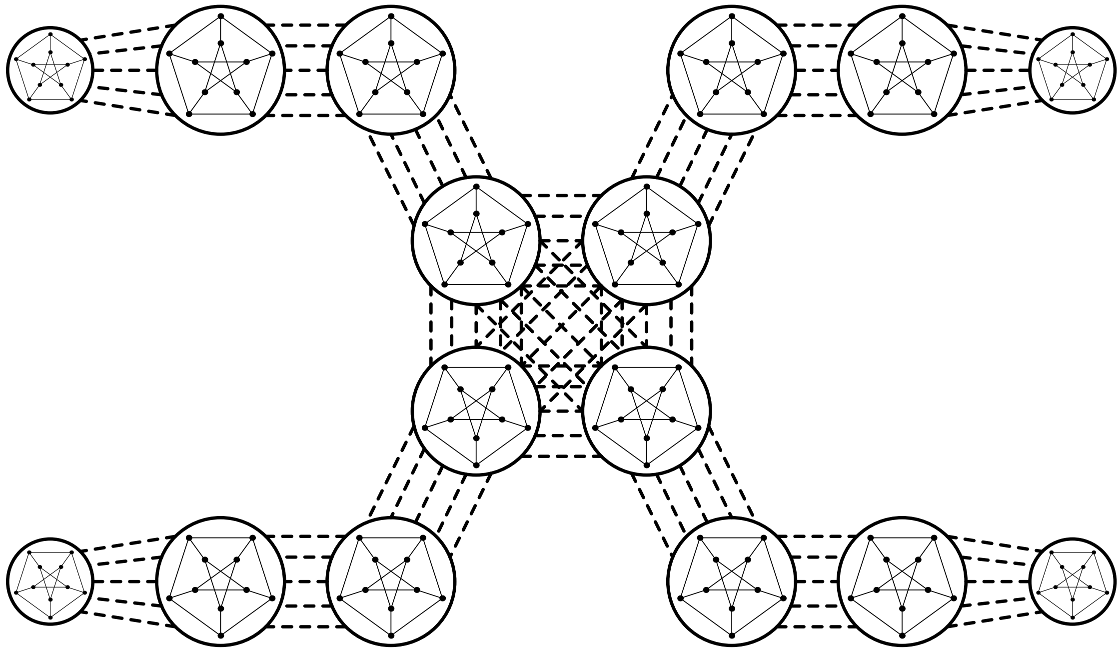

Figure 7: A sketch of the graph $U^1_{5,2,7}$ (some rays are omitted).

the special vertex form an independent set. Observe that the minimum distance for which this property does not hold is related to the odd girth of the Kneser graph $K_{p/q}$.

Proposition 2.4. *Consider a special vertex s of a universal graph $U^n_{p,q,d}$ and an integer $\ell \in \left\{1, 2, \ldots, \left\lceil \frac{q}{p-2q} \right\rceil - 1\right\}$. The vertices at distance ℓ from s form an independent set in $U^n_{p,q,d}$.*

Finally, we state the following proposition which is implicit in the proof of Theorem 1.3 in [12].

Proposition 2.5 ([12])**.** *Let $k = p/q$ be rational, where $p, q \in \mathbb{N}$ and $p \geq 2q$, $d, n \in \mathbb{N}$ and $\varepsilon > 0$. For every fractional $(k+\varepsilon)$-precoloring of the special vertices of $U^n_{p,q,d}$ by subsets $C_1, C_2, \ldots, C_{n\binom{p}{q}} \subseteq [0, k+\varepsilon)$ there exist functions f_o and f_e from $[p]$ to $2^{[0,k+\varepsilon)}$ such that the following holds:*

1) for every $i, j \in [p]$, $i \neq j$: $f_o(i) \cap f_o(j) = \varnothing$ and $f_e(i) \cap f_e(j) = \varnothing$;

2) for every $i \in [p]$ and $a \in \left[n\binom{p}{q}\right]$:

a) $\|f_o(i)\| = (k+\varepsilon)/p$ and $\|f_o(i) \cap C_a\| = 1/p$,

b) $\|f_e(i)\| = 1/q$ and $\|f_e(i) \cap C_a\| = 1/(p+q\varepsilon)$.

In other words, the function f_o in Proposition 2.5 is an equipartition of the interval $[0, k+\varepsilon)$ into p measurable parts $f_o(1), \ldots, f_o(p)$ such that the measure of the intersection of $f_o(i)$ with each set C_j, $j \in \left[n\binom{p}{q}\right]$, is the same as the expected intersection of C_j with a random subset of $[0, k+\varepsilon)$ of measure $(k+\varepsilon)/p$. Analogously, f_e is a partition of an appropriate subset of $[0, k+\varepsilon)$ of measure k into p measurable parts $f_e(1), \ldots, f_e(p)$, where the parts have measure $1/q$ and the measure of the intersection of $f_e(i)$ with each set C_j is the same as for a random subset of $[0, k+\varepsilon)$ of measure $1/q$.

3 Distances divisible by four

3.1 Upper bounds

In this section we prove upper bounds on $g(k, d)$ for $d \equiv 0 \bmod 4$ in the case that k and d satisfy $2 \le k < 2 + \dfrac{2}{d-2}$. Observe that Proposition 2.4 guarantees that if we consider the ray $R_{p,q,d/2}$, then for any $\ell \in \{1, \ldots, (d-2)/2\}$, the vertices at distance ℓ from the special vertex form an independent set.

Lemma 3.1. *Let ε be a positive real and n, p, q and d positive integers such that $d \equiv 0 \bmod 4$ and $p/q \ge 2$. If the conditions*

$$2 \le k < 2 + \frac{1}{2d'-1} \quad \text{and} \tag{1}$$

$$\varepsilon \sum_{j=0}^{d'-2} (k-1)^{2j+2} + \varepsilon \cdot \frac{k-1+\varepsilon}{k+\varepsilon} \ge \frac{1}{k+\varepsilon} \tag{2}$$

are satisfied, where $d' = d/4$ and $k = p/q$, then any fractional $(k+\varepsilon)$-precoloring of the special vertices of $U^n_{p,q,d}$ can be extended to a fractional $(k+\varepsilon)$-coloring of $U^n_{p,q,d}$.

Proof. First observe that by Proposition 1.2 we only need to consider the case that ε is the smallest positive real that satisfies inequality (2), i.e., that solves the equation

$$\varepsilon \sum_{j=0}^{d'-2} (k-1)^{2j+2} + \varepsilon \cdot \frac{k-1+\varepsilon}{k+\varepsilon} = \frac{1}{k+\varepsilon}.$$

Furthermore, it is straightforward to show that any positive solution to this equation satisfies the following two inequalities as well:

$$\varepsilon \sum_{j=0}^{d'-2} (k-1)^{2j+1} \le \frac{k-1+\varepsilon}{k+\varepsilon} \quad \text{and} \quad \varepsilon \sum_{j=0}^{d'-2} (k-1)^{2j} \le \frac{1}{k+\varepsilon}. \tag{3}$$

Now consider the universal graph $U^n_{p,q,d}$. Let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and let f_e be a mapping as described in Proposition 2.5. In what follows, for each ray R_i, which is isomorphic to $R_{p,q,2d'}$, we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 2d')$ of the base of R_i is colored by the set $f_e(A)$, and the special vertex of R_i is colored by C_i. Since the universal graph $U^n_{p,q,d}$ is constructed by identifying the vertices $(A, 2d')$, the conclusion of the lemma follows from the existence of such a fractional coloring for each ray.

Fix a ray R_i and let s be the special vertex of R_i. For an integer $\ell \in [2d' - 1]$, let V_ℓ be the set of vertices of R_i at distance ℓ from s, and let $V_{2d'}$ be the set of vertices of R_i at distance at least $2d'$ from s. Observe that the sets V_ℓ, $\ell = 1, \dots, 2d'$, form a partition of $V(R_i) \setminus \{s\}$, and if a vertex $v = (A, \ell')$ of the ray R_i is in V_ℓ, then $\ell' \leq \ell$. In particular, the vertices of the base of R_i form a subset of $V_{2d'}$. By (1) and Proposition 2.4, it follows that the set V_ℓ forms an independent set in R_i, for $\ell \in [2d' - 1]$.

The basic idea is to partition for each V_ℓ the interval $[0, k+\varepsilon)$ into three parts. The first part will be split into p equal-size parts and will be assigned to vertices in V_ℓ according to the corresponding sets in the Kneser graph. The second part will be assigned to all vertices in V_ℓ (that is possible since V_ℓ forms an independent set). The third part will not be used on the vertices of V_ℓ at all and will be reserved for the vertices in $V_{\ell-1}$. Based on the parity of V_ℓ, the second part will be either inside C_i and the third part will be disjoint from C_i, or vice versa. First we define the partition for $V_{2d'}$, and after defining the partition for some V_ℓ, we define the partition for $V_{\ell-1}$. During this procedure, the sizes of the second and third parts will increase at the expense of the first part.

Formally, we construct functions $f_x : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, $g_y : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h_z : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, for $x \in [2d']$, $y \in [d' - 1]$ and $z \in [d' - 1]$ in the following way. For $a \in [p]$ and $j = d' - 1, d' - 2, \dots, 1$, we sequentially define:

- $g_j(a)$ as an arbitrary subset of $(f_e(a) \setminus C_i) \setminus \bigcup_{j'=j+1}^{d'-1} g_{j'}(a)$ of measure $\frac{\varepsilon k}{p}(k-1)^{2(d'-j)-1}$,
- $h_j(a)$ as an arbitrary subset of $(f_e(a) \cap C_i) \setminus \bigcup_{j'=j+1}^{d'-1} h_{j'}(a)$ of measure $\frac{\varepsilon k}{p}(k-1)^{2(d'-j)-2}$,

and then:

- $f_{2d'}(a) := f_e(a)$,
- $f_{2j+1}(a) := f_{2j+2}(a) \setminus h_j(a)$, and
- $f_{2j}(a) := f_{2j+1}(a) \setminus g_j(a)$.

Finally, we set $f_1(a) := f_2(a) \setminus C_i$ for every $a \in [p]$. Since the measure of $f_e(a)$ is $1/q$ and the measure of $f_e(a) \cap C_i$ is $1/(p + q\varepsilon)$, these functions exist if and only if conditions (3) are satisfied. Next, we set Y to be the set of measure ε that is disjoint from $f_e([p])$, i.e.,

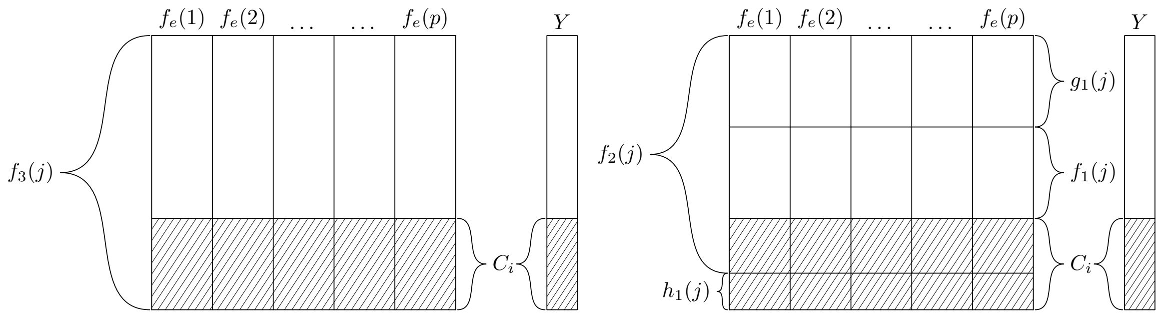

Figure 8: The construction of a fractional coloring in Lemma 3.1 for $d = 8$.

$Y := [0, k+\varepsilon) \setminus f_e([p])$. Observe that $\|Y \setminus C_i\| = \varepsilon - \varepsilon/(k+\varepsilon)$. The described construction of the functions is sketched in Figure 8.

Let $\ell \in [2d']$ and $v = (A, \ell') \in V_\ell$. Recall that $\ell' \leq \ell$. If ℓ is even, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=\ell/2}^{d'-1} h_j([p]) \, ;$$

if $\ell \geq 3$ is odd, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=(\ell+1)/2}^{d'-1} g_j([p]) \cup Y \, ;$$

and for $\ell = 1$ we set

$$c_i(v) := f_1(A) \cup \bigcup_{j=1}^{d'-1} g_j([p]) \cup (Y \setminus C_i) \, .$$

Finally, we set $c_i(s) := C_i$.

We claim that $\|c_i(v)\| \geq 1$ for every vertex $v \in V(R_i)$. Indeed, if $v = s$, then the assertion immediately follows from $\|C_i\| = 1$. Hence, in the remainder we may assume that v belongs to a set V_ℓ for some $\ell \in [2d']$. Observe that for a fixed $\ell \in [2d']$, the color sets of any two vertices u and v from V_ℓ have the same measure. Let m_ℓ be the measure of vertices in V_ℓ. Then $m_{2d'} = 1$, by the definition of f_e. If $d > 4$, then $m_{2d'-1} = m_{2d'}$, since both Y and $h_{d'-1}(A)$, for $A \in \binom{[p]}{q}$, have measure ε. Next, if $\ell \in \{3, 5, \ldots, 2d'-3\}$, then

$$m_\ell = m_{\ell+2} - \varepsilon(k-1)^{2(d'-\lfloor \ell/2 \rfloor)-2} + (k-1) \cdot \varepsilon(k-1)^{2(d'-\lceil \ell/2 \rceil)-1} = m_{\ell+2} \, .$$

Analogously, if $\ell \in \{2, 4, \ldots, 2d'-2\}$, then

$$m_\ell = m_{\ell+2} - \varepsilon(k-1)^{2(d'-\ell/2)-1} + (k-1) \cdot \varepsilon(k-1)^{2(d'-\ell/2)-2} = m_{\ell+2} \, .$$

Finally, for m_1 we have

$$m_1 = 1 - \frac{1}{k+\varepsilon} + (k-1) \cdot \varepsilon \sum_{j=1}^{d'-1} (k-1)^{2j-1} + \varepsilon - \frac{\varepsilon}{k+\varepsilon} \, ,$$

which is at least one by (2).

It remains to check that the mapping c_i assigns disjoint sets to any two adjacent vertices in R_i. Let $u = (A, \ell_u) \in V_{\ell_u}$ and $v = (B, \ell_v) \in V_{\ell_v}$ be two arbitrary adjacent vertices in R_i. Hence, A is disjoint from B and without loss of generality $\ell_u \le \ell_v \le \ell_u + 1$. If $\ell_v = \ell_u$, then $\ell_v = 2d'$ (since for $\ell < 2d'$ the set V_ℓ is independent). Thus the sets $c_i(u)$ and $c_i(v)$ are disjoint, since $f_e(A)$ and $f_e(B)$ are disjoint.

From now on, we assume that $\ell_v = \ell_u + 1$. If ℓ_v is even, then $c_i(v)$ is disjoint from Y, and disjoint from $g_j([p])$ for any $j \in \{(\ell_u+1)/2, \dots, d'-1\}$. Furthermore, $c_i(u)$ is disjoint from $h_j([p])$ for any $j \in \{\ell_v/2, \dots, d'-1\}$. Analogously if ℓ_v is odd and larger than one, then $c_i(v)$ is disjoint from $h_j([p])$ for any $j \in \{\ell_u/2, \dots, d'-1\}$, and $c_i(u)$ is disjoint from Y, and disjoint from $g_j([p])$ for any $j \in \{(\ell_v-1)/2, \dots, d'-1\}$. Since $f_\ell(A)$ is a subset of $f_e(A)$ for any $\ell \in [2d']$ and $A \in \binom{[p]}{q}$, the sets $c_i(u)$ and $c_i(v)$ are disjoint. Finally, the sets assigned to neighbors of s are disjoint from C_i.

We can conclude that the coloring c_i is a fractional coloring of the ray R_i with the required properties. $\square$

Combining Lemma 3.1 with Proposition 2.3 yields the following theorem.

Theorem 3.2. *Let d be a positive integer such that $d \equiv 0 \bmod 4$, k a rational and ε a positive real such that conditions (1) and (2) are satisfied, where $d' = \lfloor d/4 \rfloor$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least d, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

3.2 Lower bound for distance four

Let p and q be positive integers, $p/q \ge 2$, and $n = \binom{p}{q}$. Let $A = A(K_{p/q})$ be the normalized adjacency matrix of the Kneser graph $K_{p/q}$. This is the $n \times n$ matrix indexed by vertices of $K_{p/q}$ such that if $\{u, v\}$ is an edge of $K_{p/q}$, the entry corresponding to (u, v) is equal to the inverse degree of u, i.e., equal to $\binom{p-q}{q}^{-1}$, while all other entries are zero. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A such that $|\lambda_1| \ge |\lambda_2| \ge \dots \ge |\lambda_n|$, then it follows that $|\lambda_2| = \dfrac{q}{p-q}$, see [14].

A standard expansion inequality (see, e.g., [10, Theorem 4.15]) asserts that

$$|N_G(I)| \ge \frac{|I|}{(1-c)(\lambda_2)^2 + c}, \tag{4}$$

for every vertex subset I of a graph G of size at most cn, where n is the number of vertices of G. If I is an independent set of the Kneser graph $K_{p/q}$, then by the Erdős-Ko-Rado

Theorem (see, e.g., [11]), the size of I is at most $\binom{p-1}{q-1}$. Therefore, $|I|/n \leq q/p$, where $n = \binom{p}{q}$, and hence $|N(I)| \geq \dfrac{p-q}{q} \cdot |I|$ by (4). This yields the following proposition.

Proposition 3.3. *Let p and q be positive integers, $p/q \geq 2$. If I is an independent set of the Kneser graph $K_{p/q}$, then $|N(I)| \geq \dfrac{p-q}{q} \cdot |I|$.*

This proposition has a key role in proving that in any fractional k-coloring of $K_{p/q}$, where $k \geq p/q$, there is a vertex v such that the union of sets assigned to the neighborhood of v has measure at least $p/q - 1$. Note that this statement is trivial if $p/q \geq 3$, because in that case the neighborhood of any vertex of $K_{p/q}$ is isomorphic to $K_{p/q-1}$.

Lemma 3.4. *For every real $\varepsilon \geq 0$, all positive integers p and q, where $p/q \geq 2$, and any fractional $(p/q+\varepsilon)$-coloring $c : V(K_{p/q}) \to 2^{[0,p/q+\varepsilon)}$ of $K_{p/q}$, there exists a vertex $v \in V(K_{p/q})$ such that $\|c(N(v))\| \geq p/q - 1$.*

Proof. For $x \in [0, p/q+\varepsilon)$, let $V_x \subseteq V(K_{p/q})$ be the set of vertices of $K_{p/q}$ that contain x in their color set, i.e., $V_x = \{\, v \in V : x \in c(v) \,\}$. For $i \geq 1$ we define $X_i := \{\, x \in [0, p/q+\varepsilon) : |V_x| = i \,\}$. In other words, X_i are the points in $[0, k+\varepsilon)$ contained in exactly i color sets $c(v)$. Note that for $i > \binom{p-1}{q-1}$ the set X_i is empty and that $\sum_{i\geq 1} i \cdot \|X_i\| = \binom{p}{q}$. Next, let X^j be the set of points $x \in [0, k+\varepsilon)$ such that the number of vertices v that have at least one neighbor u with $x \in c(u)$ is equal to j. In other words, let $X^j := \{\, x \in [0, p/q+\varepsilon) : |N(V_x)| = j \,\}$.

Finally, consider all the intersections of X_i with X^j, where $i \in \left[\binom{p-1}{q-1}\right]$ and $j \in \left[\binom{p}{q}\right]$, and let $X_i^j := \{\, x \in [0, p/q+\varepsilon) : |V_x| = i \text{ and } |N(V_x)| = j \,\}$. Note that for a fixed j the sets X_i^j form a partition of the set X^j, where for some values of i the part X_i^j might be empty. Since for any x the set V_x forms an independent set in $K_{p/q}$, Proposition 3.3 yields that if $j < \frac{p-q}{q} \cdot i$, then X_i^j is empty. Now, for a vertex $v \in V$, consider the measure of points $x \in [0, p/q+\varepsilon)$ such that x is contained in the color set of at least one neighbor of v. By a double counting argument it follows that

$$\sum_{v\in V} \|c(N(v))\| = \sum_{j=1}^{\binom{p}{q}} j \cdot \|X^j\| = \sum_{j=1}^{\binom{p}{q}} \sum_{i=1}^{\binom{p-1}{q-1}} j \cdot \|X_i^j\| = \sum_{i=1}^{\binom{p-1}{q-1}} \sum_{j=1}^{\binom{p}{q}} j \cdot \|X_i^j\|.$$

Since the sets X_i^j are empty for $j < \frac{p-q}{q} \cdot i$, we conclude that

$$\sum_{v\in V} \|c(N(v))\| \geq \sum_{i=1}^{\binom{p-1}{q-1}} \frac{p-q}{q} \cdot i \cdot \|X_i\| = \frac{p-q}{q} \cdot \binom{p}{q}.$$

Therefore, there exists a vertex $v \in V$ such that $\|c(N(v))\| \geq p/q - 1$. ∎

We are now ready to prove that the upper bound on $g(k,4)$ for $k \in [2,3)$ given in Theorem 3.2 is best possible. The proof uses the same precoloring as was used in [12] for a lower bound in the case $k \in \{2\} \cup [3,\infty)$, but the argument for $k \in (2,3)$ is considerably more involved.

Theorem 3.5. *Let $k \in [2,3)$ be a rational number and ε a positive real such that $\varepsilon < \dfrac{1+\varepsilon}{k+\varepsilon}$. There exists a graph G with fractional chromatic number k, a subset W of its vertex set at pairwise distance at least four and a fractional $(k+\varepsilon)$-precoloring of W that cannot be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. Let ε_0 be the positive root of the equation $x = \dfrac{1+x}{k+x}$, i.e., let

$$\varepsilon_0 := \frac{1-k+\sqrt{(k-1)^2+4}}{2}.$$

Next, let p', q be positive integers such that $k+\varepsilon \le p'/q < k+\varepsilon_0$ and kq an integer. Set $\varepsilon' := p'/q - k$, $p := kq$ and $G := U^n_{p,q,4}$, where $n = \binom{p'}{q}$. We will show the existence of a $(k+\varepsilon')$-precoloring of the special vertices of G that cannot be extended to a $(k+\varepsilon')$-coloring of G. This implies that there exists also a $(k+\varepsilon)$-precoloring of the special vertices that cannot be extended to a $(k+\varepsilon)$-coloring of G by Proposition 1.2. Since the special vertices of G are at pairwise distance at least four, the statement of the theorem immediately follows.

Let $f : [p'] \hookrightarrow 2^{[0,k+\varepsilon')}$ be the function $f(i) = [(i-1)/q, i/q)$, for $i \in [p']$. Consider a precoloring of G that assigns to the n special vertices of the copies of $R^Y_{p,q,2}$, where $Y \in \binom{[p]}{q}$, all the n different sets $f(X)$, for $X \in \binom{[p']}{q}$. We claim that this fractional $(k+\varepsilon')$-precoloring cannot be extended to a fractional coloring of the whole graph.

Suppose for contradiction that there exists an extension of the precoloring given by f to a fractional coloring $c : V(G) \to 2^{[0,k+\varepsilon')}$. Let H be the base of G (that is, the common bases of all rays). Since H is isomorphic to $K_{p/q}$, Lemma 3.4 implies that there exists a vertex $v \in V(H)$ with $\|c(N_H(v))\| \ge k-1$. Let $C := c(N_H(v))$ and let u be an arbitrary neighbor of v in H; without loss of generality u is the vertex corresponding to the vertex $([q],2)$ in each ray of $U^n_{p,q,4}$.

Now consider all the rays $S^{[q]}_{p,q,2}$ in $U_{p,q,4}$; by the definition of f, for any $X \in \binom{[p']}{q}$ there is a ray where the special vertex $[q]$ is precolored with $f(X)$. Since each point of $[0,p'/q)$ is contained in exactly $\binom{p'-1}{q-1}$ sets $f(X)$, a double counting argument yields that

$$\|C\| = \frac{1}{\binom{p'-1}{q-1}} \sum_{X \in \binom{[p']}{q}} \|C \cap f(X)\|.$$

Therefore, there exists $X \in \binom{[p']}{q}$ such that $\|C \cap f(X)\| \leq \frac{q}{p'}\|C\|$. Consider the corresponding ray S with the special vertex $[q]$ precolored by $f(X)$, and let v_1 be the vertex $(v, 1)$ in S. Observe that the neighborhood of v_1 in G contains $N_H(v) \cup \{s\}$, where s is the special vertex of S. Therefore,

$$\|c(N(v_1))\| \geq \|C\| + 1 - \|C \cap f(X)\| \geq k - \frac{q}{p'}(k-1) = k - \frac{k-1}{k+\varepsilon'}\,.$$

Since $0 < \varepsilon' < \varepsilon_0$, it follows that $\varepsilon' < (1+\varepsilon')/(k+\varepsilon')$ and hence

$$k + \varepsilon' - \|c(N(v_1))\| < 1\,.$$

This implies that $c(v_1)$ intersects $c(N(v_1))$, a contradiction. □

4 Distances congruent to two mod four

4.1 Upper bounds

We start this section with showing upper bounds for $g(k, d)$, for $d \equiv 2 \bmod 4$ such that k and d satisfy $k < 2 + \frac{2}{d-2}$.

Lemma 4.1. *Let ε be a positive real and n, p, q and d positive integers such that $d \geq 6$, $d \equiv 2 \bmod 4$ and $p/q \geq 2$. If the conditions*

$$2 \leq k < 2 + \frac{1}{2d'} \quad \text{and} \tag{5}$$

$$\varepsilon \sum_{j=0}^{d'-1} (k-1)^{2j+1} \geq \frac{1}{k+\varepsilon} \tag{6}$$

are satisfied, where $d' = \lfloor d/4 \rfloor$ and $k = p/q$, then any fractional $(k+\varepsilon)$-precoloring of the special vertices of $U^n_{p,q,d}$ can be extended to a fractional $(k+\varepsilon)$-coloring of $U^n_{p,q,d}$.

Proof. As in the proof of Lemma 3.1, we only need to consider the case that ε is the positive solution of

$$\varepsilon \sum_{j=0}^{d'-1} (k-1)^{2j+1} = \frac{1}{k+\varepsilon}\,.$$

Any such solution also satisfies the following two inequalities:

$$\varepsilon \sum_{j=0}^{d'-1} (k-1)^{2j} \leq \frac{k-1+\varepsilon}{k+\varepsilon} \quad \text{and} \quad \varepsilon \sum_{j=0}^{d'-2} (k-1)^{2j+1} \leq \frac{1}{k+\varepsilon}\,. \tag{7}$$

For the universal graph $U^n_{p,q,d}$, let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and let f_e be a mapping as described in Proposition 2.5. Analogously to the proof of Lemma 3.1, for each ray R_i we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 2d'+1)$ of the base of R_i is colored by the set $f_e(A)$, and the special vertex of R_i is colored by C_i.

Fix a ray R_i and let s be the special vertex of R_i. For an integer $\ell \in [2d']$, let $V_\ell \subseteq V(R_i)$ be the set of vertices at distance ℓ from s, and let $V_{2d'+1}$ be the set of vertices of R_i at distance at least $2d'+1$ from s. Similarly to the proof of Lemma 3.1, the vertices of the base of R_i form a subset of $V_{2d'+1}$, and the set V_ℓ forms an independent set in R_i, for $\ell \in [2d']$.

We construct functions $f_x : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, $g_y : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h_z : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, for $x \in [2d'+1]$, $y \in [d']$ and $z \in [d'-1]$ as follows. For $a \in [p]$ and $j = d'-1, d'-2, \dots 1$ we sequentially define:

- $h_j(a)$ as an arbitrary subset of $(f_o(a) \cap C_i) \setminus \bigcup_{j'=j+1}^{d'-1} h_{j'}(a)$ of measure $\frac{\varepsilon k}{p}(k-1)^{2(d'-j)-1}$,

Next, we sequentially define for $a \in [p]$ and $m = d', d'-1, \dots, 1$

- $g_m(a)$ as an arbitrary subset of $(f_e(a) \setminus C_i) \setminus \bigcup_{m'=m+1}^{d'} g_{m'}(a)$ of measure $\frac{\varepsilon k}{p}(k-1)^{2(d'-m)}$,

and then:

- $f_{2d'+1}(a) := f_e(a)$,
- $f_{2m+1}(a) := f_{2m+2}(a) \setminus h_m(a)$ for $m < d'$, and
- $f_{2m}(a) := f_{2m+1}(a) \setminus g_m(a)$.

Finally, we set $f_1(a) := f_2(a) \setminus C_i$ and $Y := [0, k+\varepsilon) \setminus f_e([p])$. Similarly as in the proof of Lemma 3.1, such functions exist if and only if conditions (7) are satisfied. The described construction of the functions is sketched in Figure 9.

Let $\ell \in [2d']$ and $v = (A, \ell') \in V_\ell$. If ℓ is even, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=\ell/2}^{d'-1} h_j([p]) \cup Y\,;$$

and if ℓ is odd, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=(\ell+1)/2}^{d'-1} g_j([p])\,.$$

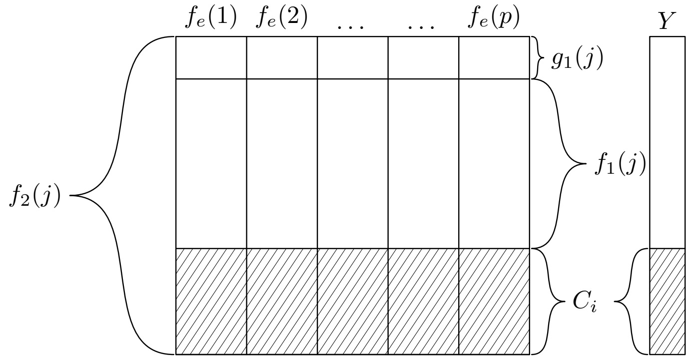

Figure 9: The construction of a fractional coloring in Lemma 4.1 for $d = 6$.

Setting $c_i(s) := C_i$, together with an analysis analogous to the that presented in the proof of Lemma 3.1, yield that c_i is a fractional coloring of the ray R_i with the required properties. □

Combining the lemma with Proposition 2.3 yields the following theorem.

Theorem 4.2. *Let d be an integer such that $d \geq 6$ and $d \equiv 2 \bmod 4$, k a rational and ε a positive real such that conditions (5) and (6) are satisfied, where $d' = \lfloor d/4 \rfloor$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least d, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

We close this section by showing an upper bound on $g(k, 6)$ for $k \in [2.5, 3)$, which is best possible due to Theorem 4.5.

Theorem 4.3. *Let k be a positive rational less than 3 and ε a positive real such that $\varepsilon \geq \dfrac{1}{k+\varepsilon}$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least six, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. By Proposition 2.3, it is enough to consider only the universal graphs $U^n_{p,q,6}$, where $p/q = k$ and $n \in \mathbb{N}$, and an arbitrary precoloring of its special vertices. As in the proofs of Lemmas 3.1 and 4.1, let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and let f_e be a mapping as described in Proposition 2.5. For each ray R_i we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 3)$ of the base of R_i is colored by the set $f_e(A)$, and the special vertex of R_i is colored by C_i.

Fix a ray R_i, let s be the special vertex of R_i and set $Y := [0, k+\varepsilon) \setminus f_e([p])$. By symmetry, it is enough to consider the case where R_i is a copy of $R^{[q]}_{p,q,3}$. We construct functions $g : [q] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h : [q] \hookrightarrow 2^{[0,k+\varepsilon)}$ as follows. For $j \in [q]$ we define $g(j)$ to

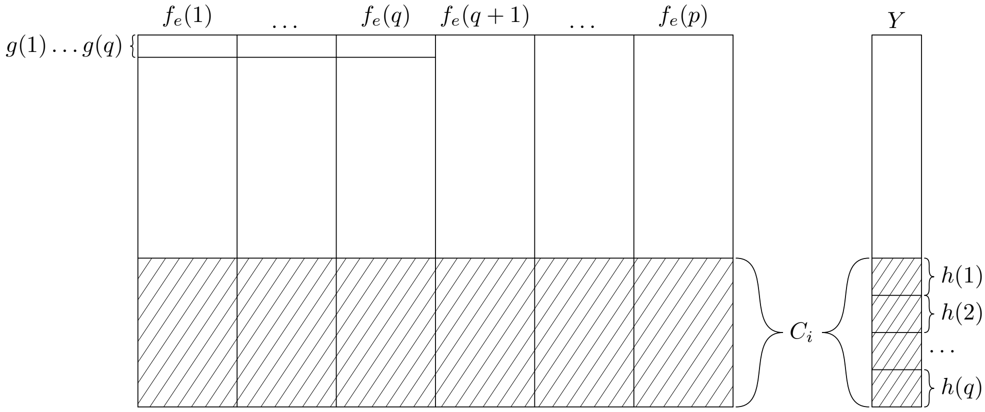

Figure 10: The construction of a fractional coloring in Theorem 4.3.

be an arbitrary subset of $f_e(j) \setminus C_i$ of measure $\dfrac{\varepsilon}{p+q\varepsilon}$. Note that these subsets always exist since $\|f_e(j) \setminus C_i\| = \dfrac{k-1+\varepsilon}{p+q\varepsilon}$. Next, we define $h(1), h(2), \dots, h(q)$ as an arbitrary equipartition of $Y \cap C_i$ into q parts of measure $\dfrac{\varepsilon}{p+q\varepsilon}$. The described construction of the functions is sketched in Figure 10.

Recall that the neighborhood of s in R_i forms an independent set. Since $s = ([q], 0)$, for every neighbor (A, ℓ') of s we have $A \cap [q] = \varnothing$ and $\ell' = 1$. We now construct a fractional coloring of R_i. Let $v = (A, \ell')$ be a vertex of R_i and let ℓ be the distance of v from s in R_i. We define $c_i(v)$ in the following way:

- if $\ell \geq 3$, then $c_i(v) := f_e(A)$;
- if $\ell = 2$, then $c_i(v) := (f_e(A) \setminus g(A \cap [q])) \cup h(A \cap [q])$;
- if $\ell = 1$, then $c_i(v) := (f_e(A) \setminus C_i) \cup (Y \setminus C_i) \cup g([q])$; and
- $c_i(s) := C_i$.

It is straightforward to check that we assigned disjoint sets to any two neighbors in R_i, and that any vertex at distance at least two from s is assigned a set of measure one. Furthermore, for every $A \in \binom{[p] \setminus [q]}{q}$ the set $f_e(A) \setminus C_i$ is disjoint from both Y and $g([q])$, and has measure $1 - \dfrac{1}{k+\varepsilon}$. Since $(Y \setminus C_i) \cup g([q])$ has measure $\varepsilon \geq \dfrac{1}{k+\varepsilon}$ (recall that $\|Y \cap C_i\| = \varepsilon/(k+\varepsilon)$), it follows that c_i is a fractional coloring with the required properties. $\square$

4.2 Lower bound for distance six

The goal of this section is to prove that $g(k, 6) = \frac{1}{2}\left(\sqrt{k^2+4} - k\right)$ for $k \in [2.5, 3)$, i.e., $g(k, 6)$ is the positive root of the equation $x = \dfrac{1}{k+x}$ for k in that range. Before we present a formal proof, let us first sketch the idea. Suppose for a contradiction that there exists $k \in [2.5, 3)$ and $\varepsilon > 0$ such that $g(k, 6) \leq \varepsilon$ and $\varepsilon < \dfrac{1}{k+\varepsilon}$. As in the proof of Theorem 3.5, we may assume that ε is a rational. Let p' and q be integers such that $p'/q = k + \varepsilon$ and kq is an integer, and let $p = kq$. We construct a precoloring of the special vertices of $U^n_{p,q,6}$, where $n = \binom{p'}{q}$, such that each point of the interval $[0, k + \varepsilon)$ belongs to exactly $\binom{p}{q} \cdot \binom{p'-1}{q-1}$ sets.

Now fix an arbitrary extension $c : V(U^n_{p,q,6}) \to 2^{[0,k+\varepsilon)}$ of this precoloring to a fractional coloring of $U^n_{p,q,6}$, and let c' be the restriction of c to the special vertices and the common bases of the rays. For every ray R_i of $U^n_{p,q,6}$, we will consider a linear program P_i that minimizes the value of a fractional coloring that extends c'. Clearly, for every ray R_i the optimal solution of P_i has value at most $k+\varepsilon$. On the other hand, we will show that there is a ray R_i such that the optimum of the dual program to P_i is at least $k + \dfrac{1}{k+\varepsilon}$. Therefore, by weak duality of linear programming (see e.g. [16]), it follows that $k + \dfrac{1}{k+\varepsilon} \leq k + \varepsilon$, which contradicts the assumption.

We start the formal exposition by introducing the notion dual to fractional colorings. Let $G = (V, E)$ be a graph. We say that a mapping $x : V(G) \to [0, 1]$ is a *fractional clique* in G if for every independent set I of G the sum $\sum_{v \in I} x(v)$ is at most one. The *weight of* x is the sum $\sum_{v \in V(G)} x(v)$. The problem of determining the maximum weight of a fractional clique in G can be also formulated as a linear program. This program is the dual program to the program that determines the fractional chromatic number of G. For a fractional clique x in a graph G and a vertex subset $S \subseteq V(G)$ we set $x(S) := \sum_{v \in S} x(v)$.

The following proposition asserts that there exists a fractional clique of weight p/q in the Kneser graph $K_{p/q}$, where $p/q \in [2.5, 3)$ and q is even, such that there exists a neighborhood $N(v)$, where the sum of the weights over $N(v)$ is equal to one.

Proposition 4.4. *For every positive integer p and for every positive even integer q such that $p/q \in [2.5, 3)$, there exists a fractional clique $x : \binom{[p]}{q} \to [0, 1]$ in $K_{p/q}$ of weight p/q such that $x(v) = \binom{p-q}{q}^{-1}$ for every neighbor v of the vertex $[q]$.*

Proof. Let $V_0 := \{[q]\}$, $V_1 := \binom{[p] \setminus [q]}{q}$ and $V_2 := \{ X \in \binom{[p]}{q} : |X \cap [q]| = q/2 \}$ be vertex subsets of $K_{p/q}$. Note that $|V_1| = \binom{p-q}{q}$, $|V_2| = \binom{q}{q/2} \cdot \binom{p-q}{q/2}$, and the sets V_0, V_1 and V_2 are pairwise disjoint. Let H be the subgraph of $K_{p/q}$ induced by $V_0 \cup V_1 \cup V_2$. Observe that H is connected since $p/q \geq 2.5$. We will show the existence of a fractional clique x in H of weight k such that $x(v_1) = |V_1|^{-1}$ for each vertex $v_1 \in V_1$. The statement of the proposition then follows.

For each vertex $v \in V(H)$, define

$$x(v) := \begin{cases} 3 - p/q, & \text{if } v = [q]; \\ \dfrac{1}{|V_1|}, & \text{if } v \in V_1; \\ \dfrac{2(p/q-2)}{|V_2|}, & \text{if } v \in V_2. \end{cases}$$

The weight of $x(H)$ is equal to p/q. Hence, it remains to check that $x(I) \leq 1$ for every independent set I of H.

Fix an independent set I of H. First suppose that $[q] \in I$. Observe that since I has to be disjoint from V_1, it is enough to show that $|I \cap V_2| \leq |V_2|/2$. Consider the subgraph H' induced by V_2. Since every vertex in H' has degree $\binom{p-3q/2}{q/2}$, the graph H' is regular and therefore it has independence number at most $|V(H')|/2$.

In the remainder of the proof we suppose that $[q] \notin I$. Next set $S_1 := I \cap V_1$, $S_2 := \{ X \in \binom{[p]\setminus[q]}{q/2} : \text{there is a } Y \in I \cap V_2 \text{ with } X \subseteq Y \}$, and $S := S_1 \cup S_2$. If S_2 is empty, then I is a subset of V_1, and therefore $x(I)$ is at most one. On the other hand, if S_1 is empty, then I is a subset of V_2. As the graph induced by V_2 is $\binom{p-3q/2}{q/2}$-regular, hence $x(I) \leq p/q - 2 < 1$.

So we can assume that both S_1 and S_2 are non-empty. For a set $X \in S_2$, we define $\widehat{x}(X) := \sum_{Y \in I \cap V_2 \text{ s.t. } X \subseteq Y} x(Y)$. Now let $\mathcal{O}$ be the set of all $(p-q-1)!$ circular orders of the set $[p] \setminus [q]$. We say that a set $Z \subseteq [p] \setminus [q]$ is an *arc* in $O \in \mathcal{O}$ if we can order the elements of Z in such a way that they form a consecutive segment in O. For every $O \in \mathcal{O}$, we define the set S^O as the subset of S that contains $X \in S$ if and only if X is an arc in O.

Analogously, define S_1^O as the family of sets $X \in S_1$ that are arcs in O, and S_2^O as the family of sets $X \in S_2$ that are arcs in O. Observe that for every $X \in S_1$ there exists $q!(p-2q)!$ choices of O such that $X \in S_1^O$, and for every $X \in S_2$ there exists $(q/2)!(p-3q/2)!$ choices of O such that $X \in S_2^O$. Consider the function $x' : S \to [0,1]$ defined as follows:

- for $X \in S_1$, set $x'(X) := \dfrac{1}{p-q} = \dfrac{(p-q-1)!}{q!(p-2q)!} \cdot x(X)$; and
- for $X \in S_2$, set $x'(X) := \dfrac{2(p/q-2) \cdot |\{ Y \in I : X \subseteq Y \}|}{(p-q) \cdot \binom{q}{q/2}} = \dfrac{(p-q-1)!}{(q/2)!(p-3q/2)!} \cdot \widehat{x}(X)$.

By a double counting argument,

$$(p-q-1)! \cdot x(I) = \sum_{O \in \mathcal{O}} \sum_{X \in S^O} x'(X)\,.$$

Therefore, it is enough to show that for every $O \in \mathcal{O}$ the sum $\sum_{X \in S^O} x'(X)$ is at most one. Let $x'(O)$ be this sum.

Fix a circular order $O \in \mathcal{O}$. If S_2^O is empty, then $x'(O) = \frac{|S_1^O|}{p-q} \le 1$. If S_1^O is empty, then we show that $x'(O) \le p/q - 2$. Indeed, consider the subgraph H_O of H induced by $A \cup X$, where $A \in \binom{[q]}{q/2}$ and $X \in S_2$. Note that $|V(H_O)| \le (p-q)\binom{q}{q/2}$. By the definitions of x' and $\widehat{x}$,

$$x'(O) = \frac{\binom{p-q}{q/2}}{(p-q)} \cdot x(I \cap V(H_O)) = 2(p/q-2) \cdot \frac{|I \cap V(H_O)|}{(p-q)\binom{q}{q/2}} .$$

Since the graph H_O is $(p-2q+1)$-regular, $|I \cap V(H_O)| \le |V(H_O)|/2 \le (p-q)\binom{q}{q/2}/2$, and hence $x'(O) \le p/q - 2$.

Finally, consider the case that both S_1^O and S_2^O are non-empty. We claim that $|S^O| \le 3q/2$. We say that an arc L in O of size $q/2$ is *forbidden* for S_1^O if there exists a set in S_1^O that is disjoint from L. Let $s_1 = |S_1^O|$ and $s_2 = |S_2^O|$. Every arc in O of size $q/2$ intersects at most $3q/2 - 1$ arcs in O of size q, hence $s_1 \le 3q/2 - 1$. On the other hand, we show that at least $p - 5q/2 + s_1$ arcs in O of size $q/2$ are forbidden for S_1^O, which means that $s_2 \le 3q/2 - s_1$.

Fix an arbitrary cyclic numbering of the elements of the set $[p] \setminus [q]$ with numbers $1, 2, \ldots, p-q$ such that any two consecutive elements in O have consecutive numbers. Let K_ℓ, for $\ell \in [p-q]$, be the arc in O of size q that starts at the ℓ-th element of O and contains the next $q-1$ elements of O. Analogously, let L_ℓ, for $\ell \in [p-q]$, be the arc of size $q/2$ that starts at the ℓ-th element and contains the next $q/2-1$ elements. For brevity, we also refer to K_{p-q} as K_0, and to L_{p-q} as L_0. If the sets in S_1^O correspond to s_1 consecutive arcs, i.e., for a fixed $\ell \in [p-q]$ the set S_1^O is equal to $\{ K_{\ell+j \bmod p-q} : j = 0, \ldots, s_1 - 1 \}$, then observe that exactly $p - 5q/2 + s_1$ arcs in O are forbidden for S_1^O.

Suppose now that the sets in S_1^O do not correspond to s_1 consecutive arcs. By symmetry, we may assume that $K_1 \in S_1^O$, $K_j \in S_1^O$ for some $j \in \{3, \ldots, p-q-1\}$, and $K_{j'} \notin S_1^O$ for every $j' = 2, \ldots, j-1$. We will show that there exists a set T of s_1 consecutive arcs in O of size q such that the number of forbidden arcs in O of size $q/2$ for S_1^O is at least the number of forbidden arcs for T. If the arc $L_{p-3q/2+2}$, i.e., the arc that ends at the first element of O, is disjoint from K_j, then every set that is disjoint from K_2 is also disjoint from K_1 or K_j. Therefore, every arc in O of size $q/2$ that is forbidden for $T' := (S_1^O \setminus \{K_j\}) \cup \{K_2\}$ is also forbidden for S_1^O.

If the arc $L_{p-3q/2+2}$ intersects K_j, then since $K_{j'}$ is not in S_1^O for all $j' = 2, \ldots, j-1$, the arc $L_{j+q \bmod p-q}$ is disjoint from K_j and intersects every other set in S_1^O. Since it intersects also K_2, the number of forbidden arcs in O of size $q/2$ for $T' := (S_1^O \setminus \{K_j\}) \cup \{K_2\}$ is at most the number of forbidden arcs for S_1^O. By repeating this procedure till the arcs in O in the set T' are consecutive, we conclude that the number of forbidden arcs for S_1^O is at least $p - 5q/2 + s_1$.

Now if $s_1 \geq q$, then

$$x'(O) = \frac{s_1}{p-q} + \sum_{X \in S_2^O} x'(X) \leq \frac{s_1 + (3q/2 - s_1) \cdot 2(p/q - 2)}{p-q} \leq 1\,,$$

since the nominator of the last fraction is equal to $3p - 6q - s_1(2 \cdot p/q - 5)$, which is at most $p - q$.

On the other hand, if $s_1 < q$, then consider the partition of the set $\binom{[q]}{q/2}$ into $\binom{q}{q/2}/2$ unordered pairs $\{A, B\}$ such that A and B are disjoint. Fix such a pair $\{A, B\}$. We claim that the number of tuples (L, Z), where $L \in S_2^O$, $Z \in \{A, B\}$ and $L \cup Z \in I$, is at most $q/2 + s_2$. Indeed, otherwise there would be at least $q/2 + 1$ arcs $L \in S_2^O$ such that both $L \cup A$ and $L \cup B$ are in I. Since every arc in O of size $q/2$ intersects $q/2 - 1$ other arcs of size $q/2$, there exist two disjoint sets in I, which contradicts the fact that I is an independent set. Therefore, it follows that

$$\sum_{X \in S_2^O} x'(X) \leq \frac{(q/2 + s_2)(p/q - 2)}{(p-q)}$$

and

$$x'(O) = \frac{s_1}{p-q} + \sum_{X \in S_2^O} x'(X) \leq \frac{s_1 + (q/2 + 3q/2 - s_1)(p/q - 2)}{p-q} < 1\,.$$

The last inequality holds since the nominator of the last fraction is equal to $s_1(3 - p/q) + 2p - 4q$, which is less than $p - q$. □

We are now ready to give a lower bound on $g(k, 6)$ for $k \in [2.5, 3)$.

Theorem 4.5. *Let $k \in [2.5, 3)$ be a rational number and ε a positive real such that $\varepsilon < \frac{1}{k+\varepsilon}$. There exists a graph G with fractional chromatic number k, a subset W of its vertex set at pairwise distance at least six and a fractional $(k+\varepsilon)$-precoloring of W that cannot be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. As in the proof of Theorem 3.5, let ε_0 be the positive root of the equation $x = \frac{1}{k+x}$ and let p' and q be positive integers such that q is even, $k + \varepsilon \leq p'/q < k + \varepsilon_0$ and kq is an integer. Next, set $p := kq$, $\varepsilon' := p'/q - k$ and $G := U^n_{p,q,6}$, where $n = \binom{p'}{q}$. We will show the existence of a fractional $(k+\varepsilon')$-precoloring of the special vertices of G that cannot be extended to a fractional $(k+\varepsilon')$-coloring of G, which implies the statement of the theorem by Proposition 1.2.

Let $f : [p'] \hookrightarrow 2^{[0,k+\varepsilon')}$ be the function $f(i) = [(i-1)/q, i/q)$, for $i \in [p']$. Consider a precoloring of G that assigns to the n special vertices of the copies of $R^Y_{p,q,3}$, $Y \in \binom{[p]}{q}$, all the n different sets $f(X)$, for $X \in \binom{[p']}{q}$. We assert that this fractional $(k+\varepsilon')$-precoloring cannot be extended to a fractional coloring of the whole graph.

Suppose, on the contrary, that there exists an extension of the precoloring given by f to a fractional coloring $c : V(G) \to 2^{[0,k+\varepsilon')}$. Let H be the base of G (recall that H is isomorphic to $K_{p/q}$), let $\mathcal{I}_H$ be the set of all independent sets in H, and for every $I \in \mathcal{I}_H$ let $\overline{I}$ be the complement of I in H, i.e., $\overline{I} = V(H) \setminus I$.

For every ray R_i with its special vertex colored by C_i and every independent set $I \in \mathcal{I}_H$, let $d_i(I)$ be the measure of the set of all points in $[0, k+\varepsilon') \cap C_i$ assigned by c to all vertices in I and none in $\overline{I}$. In other words,

$$d_i(I) := \left\| \bigcap_{v \in I} (c(v) \cap C_i) \setminus c(\overline{I}) \right\| .$$

Analogously, let $e_i(I)$ be the measure of points of $[0, k+\varepsilon') \setminus C_i$ used in the coloring of H with c exactly on vertices of I, i.e.,

$$e_i(I) := \left\| \bigcap_{v \in I} (c(v) \setminus C_i) \setminus c(\overline{I}) \right\| .$$

Finally, set $t_i(I) := d_i(I) + e_i(I)$. Observe that for every vertex $v \in V(H)$ the sum of $t_i(I)$ over all independent sets I that contain v is equal to one.

Now let N be the neighborhood in H of the vertex $[q]$. (Where $[q]$ is the vertex obtained by identifying the vertices $([q], 3)$ from all rays R_i.) Recall that $|N| = \binom{p-q}{q}$. We assert that there exists a ray R_i with the special vertex $([q], 0)$ for which

$$\sum_{I \in \mathcal{I}_H} |N \cap I| \cdot d_i(I) \geq |N| \cdot \frac{1}{k+\varepsilon'} = \binom{p-q}{q} \cdot \frac{q}{p'} . \tag{8}$$

Indeed, let $C_1, C_2, \ldots, C_n$ be the sets used in the precoloring of the vertex $([q], 0)$ in the rays $R^{[q]}_{p,q,3}$. For simplicity, let $R_1, R_2, \ldots, R_n$ be these rays and $d_1, e_1, \ldots, d_n, e_n$ are the corresponding functions defined above. Since each point of $[0, k+\varepsilon')$ is contained in exactly $\binom{p'-1}{q'-1}$ sets C_i, it follows that $\sum_{i=1}^{n} d_i(I) = \binom{p'-1}{q'-1} \cdot t_i(I)$ for every $I \in \mathcal{I}_H$. Next, by a double counting argument we have $\sum_{I \in I_H} |I \cap N| \cdot t_i(I) = |N| = \binom{p-q}{q}$. Therefore,

$$\sum_{i=1}^{n} \sum_{I \in \mathcal{I}_H} |N \cap I| \cdot d_i(I) = \sum_{I \in \mathcal{I}_H} |N \cap I| \left(\sum_{i=1}^{n} d_i(I) \right) = \binom{p'-1}{q-1} \cdot \binom{p-q}{q} .$$

Since $n = \binom{p'}{q}$, there exists a ray R_i with the special vertex $([q], 0)$ such that inequality (8) holds. In the remainder of the proof, we fix R_i to be such a ray and let s be the special vertex in R_i.

Let $\mathcal{I}_R$ be the set of all independent sets in the ray R_i and let $V' := V(R_i)\setminus(V(H)\cup\{s\})$. Consider the following linear program P:

$$\begin{aligned}
\text{minimize:}\quad & \sum_{I\in\mathcal{I}_R} w(I); && \\
\text{subject to:}\quad & \sum_{I\in\mathcal{I}_R,\, v\in I} w(I) \geq 1, && \forall v \in V'; \\
& \sum_{\substack{I\in\mathcal{I}_R,\, s\in I\\ I\cap H = I_H}} w(I) \geq d_i(I_H), && \forall I_H \in \mathcal{I}_H; \\
& \sum_{\substack{I\in\mathcal{I}_R,\, s\notin I\\ I\cap H = I_H}} w(I) \geq e_i(I_H), && \forall I_H \in \mathcal{I}_H; \\
& w(I) \geq 0, && \forall I \in \mathcal{I}_R.
\end{aligned}$$

Observe that the fact that c is a fractional $(k+\varepsilon')$-coloring of G implies that there exists a solution satisfying the conditions of P such that $\sum_{I\in\mathcal{I}_R} w(I) \leq k+\varepsilon'$. Now consider the dual program P^* of P:

$$\begin{aligned}
\text{maximize:}\quad & \sum_{v\in V'} y(v) + \sum_{I\in\mathcal{I}_H} \left[d_i(I)\cdot y^d(I) + e_i(I)\cdot y^e(I)\right]; && \\
\text{subject to:}\quad & y^d(I\cap H) + \sum_{v\in I} y(v) \leq 1, && \forall I \in \mathcal{I}_R \text{ s.t. } s \in I; \\
& y^e(I\cap H) + \sum_{v\in I} y(v) \leq 1, && \forall I \in \mathcal{I}_R \text{ s.t. } s \notin I; \\
& y(v) \geq 0, && \forall v \in V'; \\
& y^d(I) \geq 0,\ y^e(I) \geq 0, && \forall I \in \mathcal{I}_H.
\end{aligned}$$

We will show that there exists a feasible solution of P^* such that the objective function of P^* is at least $k + q/p' = k + \dfrac{1}{k+\varepsilon'}$. Therefore, $\varepsilon' \geq \dfrac{1}{k+\varepsilon'}$, which is a contradiction with the choice of ε'.

Let $x : \binom{[p]}{q} \to [0,1]$ be a fractional clique in $K_{p/q}$ of weight p/q such that for every $X \in \binom{[p]\setminus[q]}{q}$ we have $x(X) = \binom{p-q}{q}^{-1}$. Proposition 4.4 implies that such a clique exists (recall that q is even). We now define an embedding g of $K_{p/q}$ in the subgraph of R_i induced by V'. If $X \in \binom{[p]}{q}$ is disjoint from $[q]$, we set $g(X) := (X, 1)$; otherwise we set

$g(X) := (X, 2)$. For every set $X \in \binom{[p]}{q}$ we set $y(g(X)) := x(X)$, and for every other vertex $v \in V' \setminus g\left(\binom{[p]}{q}\right)$ we set $y(v) := 0$. Finally, we set $y^d(I) := \dfrac{|N \cap I|}{\binom{p-q}{q}}$ and $y^e(I) := 0$ for every $I \in \mathcal{I}_H$.

By the definition of x, it follows that $\sum_{v \in V'} y(v) = k$. Next, inequality (8) implies that $\sum_{I \in \mathcal{I}_H} d_i(I) \cdot y^d(I) \geq q/p'$. Since $y(v)$, for $v \in V'$, forms a fractional clique in R_i, it remains to show that for $I \in \mathcal{I}_R$, where $s \in I$ and $I \cap N \neq \varnothing$, we have $y^d(I \cap H) + \sum_{v \in I} y(v) \leq 1$. We show that for every I, where $s \in I$ and $I \cap N \neq \varnothing$, there exists an independent set $I' \in \mathcal{I}_R$ that is disjoint from N and that satisfies

$$y^d(I \cap H) + \sum_{v \in I} y(v) = \sum_{v \in I'} y(v)\,. \tag{9}$$

Since $y(v)$ is a fractional clique in R_i and I' is an independent set, it follows that $\sum_{v \in I'} y(v) \leq 1$. We construct I' in the following way:

- I' is disjoint from $\left\{ (X, 0) : X \in \binom{[p]}{q} \right\} \cup \left\{ (X, 3) : X \in \binom{[p]}{q} \right\}$,
- $(X, 1) \in I'$ if and only if $(X, 3) \in I \cap N$ (observe that $(X, 1) \notin I$), and
- $(X, 2) \in I'$ if and only if $(X, 2) \in I$.

By the choice of $y^d(H \cap I)$, and since $y((X, 1)) = \binom{p-q}{q}^{-1}$ for $X \in \binom{[p] \setminus [q]}{q}$, it follows that equation (9) holds. This completes the proof. □

5 Distances congruent to one mod four

Analogously to Section 3.1, in this section we present upper bounds on $g(k, d)$ for $d \equiv 1 \bmod 4$ in the case that k and d satisfy $2 \leq k < 2 + \dfrac{2}{d-3}$.

Lemma 5.1. *Let ε be a positive real and n, p, q and d positive integers such that $d \geq 5$, $d \equiv 1 \bmod 4$ and $p/q \geq 2$. If the conditions*

$$2 \leq k < 2 + \frac{1}{2d'-1} \quad \textit{and} \tag{10}$$

$$\varepsilon k \sum_{j=0}^{d'-1} (k-1)^{2j} \geq 1 \tag{11}$$

are satisfied, where $d' = \lfloor d/4 \rfloor$ and $k = p/q$, then any fractional $(k+\varepsilon)$-precoloring of the special vertices of $U^n_{p,q,d}$ can be extended to a fractional $(k+\varepsilon)$-coloring of $U^n_{p,q,d}$.

Proof. Again, we only need to consider ε that satisfy (11) with equality, i.e., we can take

$$\varepsilon = \left(k \sum_{j=0}^{d'-1} (k-1)^{2j} \right)^{-1}.$$

Note that trivially gives

$$\varepsilon k \sum_{j=0}^{d'-2} (k-1)^{2j} \le 1. \tag{12}$$

Considering the universal graph $U^n_{p,q,d}$, let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and let f_o be a mapping as described in Proposition 2.5. In what follows, for each ray R_i, which is isomorphic to $R_{p,q,2d'}$, we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 2d')$ of the base of R_i is colored by a subset of $f_o(A)$, and the special vertex of R_i is colored by C_i. Since the universal graph $U^n_{p,q,d}$ is constructed by joining the vertices $(A, 2d')$ and $(B, 2d')$ from different rays for disjoint $A, B \in \binom{[p]}{q}$, the lemma follows from this claim.

Fix a ray R_i and let s be the special vertex of R_i. For an integer $\ell \in [2d'-1]$, let V_ℓ be the set of vertices of R_i at distance ℓ from s, and let $V_{2d'}$ be the set of vertices of R_i at distance at least $2d'$ from s. As in the proof of Lemma 3.1, the vertices of the base of R_i form a subset of $V_{2d'}$ and the set V_ℓ forms an independent set in R_i for $\ell \in [2d'-1]$.

Analogously to the proof of Lemma 3.1, we construct functions $f_x : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, $g_y : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h_z : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, for $x \in [2d']$, $y \in [d']$ and $z \in [d'-1]$ as follows. For $a \in [p]$ and $j = d'-1, d'-2, \ldots, 1$, we sequentially define

- $g_{d'}(a)$ as an arbitrary subset of $f_o(a) \setminus C_i$ of measure $\dfrac{\varepsilon}{p}$,
- $g_j(a)$ as an arbitrary subset of $(f_o(a) \setminus C_i) \setminus \bigcup_{j'=j+1}^{d'} g_{j'}(a)$ of measure $\dfrac{\varepsilon k}{p}(k-1)^{2(d'-j)-1}$,
- $h_j(a)$ as an arbitrary subset of $(f_o(a) \cap C_i) \setminus \bigcup_{j'=j+1}^{d'-1} h_{j'}(a)$ of measure $\dfrac{\varepsilon k}{p}(k-1)^{2(d'-j)-2}$,

and then:

- $f_{2d'}(a) := f_o(a) \setminus g_{d'}(a)$,
- $f_{2j+1}(a) := f_{2j+2}(a) \setminus h_j(a)$, and
- $f_{2j}(a) := f_{2j+1}(a) \setminus g_j(a)$.

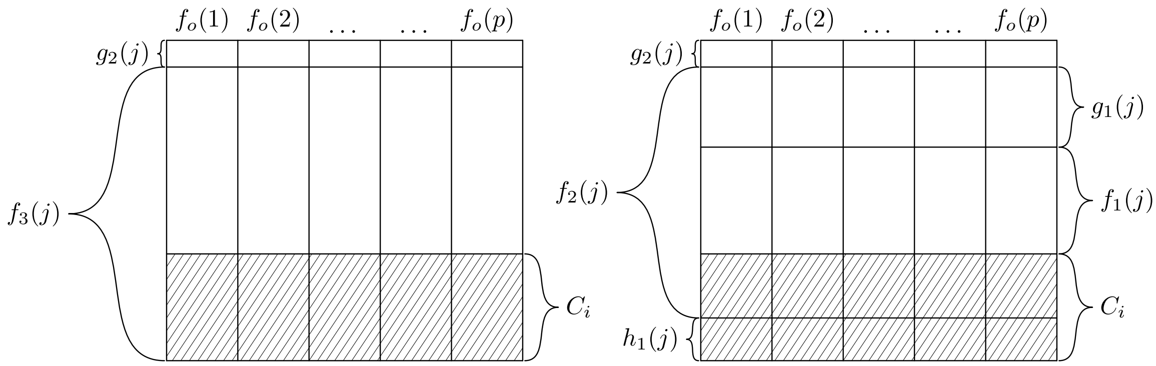

Figure 11: The construction of a fractional coloring in Lemma 5.1 for $d = 9$.

Finally, we set $f_1(a) := f_2(a) \setminus C_i$ for every $a \in [p]$. Since the measure of $f_o(a)$ is $(k+\varepsilon)/p$ and the measure of $f_o(a) \cap C_i$ is $1/p$, these functions exist if and only if condition (12) is satisfied. The described construction of the functions is sketched in Figure 11.

Let $\ell \in [2d']$ and $v = (A, \ell') \in V_\ell$. Recall that $\ell' \leq \ell$. If ℓ is even, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=\ell/2}^{d'-1} h_j([p])\,;$$

and if ℓ is odd, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=(\ell+1)/2}^{d'} g_j([p])\,.$$

Finally, we set $c_i(s) := C_i$.

As in the proof of Lemma 3.1, we claim that $\|c_i(v)\| \geq 1$ for every vertex $v \in V(R_i)$. First, it follows from the definition that $\|c_i(s)\| = 1$. Next, for a fixed $\ell \in [2d']$, the color sets of any two vertices u and v from V_ℓ have the same measure. Let m_ℓ be the measure of vertices in V_ℓ. Then $m_{2d'} = \dfrac{k+\varepsilon}{k} - \dfrac{\varepsilon}{k} = 1$. Next, for $\ell \in \{2, 3, \ldots, 2d'-1\}$ we have $m_\ell = 1$, by analogous calculations as in the proof of Lemma 3.1. Finally, for m_1 we obtain

$$m_1 = 1 - \frac{1}{k} - (k-1)\cdot \varepsilon k \sum_{j=1}^{d'-1} (k-1)^{2j-1} + \varepsilon = 1 - \frac{1}{k} + \varepsilon \sum_{j=0}^{d'-1} (k-1)^{2j}\,,$$

which is at least one by (11).

An analysis analogous to that presented in the proof of Lemma 3.1 yields that c_i assigns disjoint sets to any two adjacent vertices in R_i. Therefore, the coloring c_i is a fractional coloring of the ray R_i with the required properties. $\square$

As in Section 3.1, applying Proposition 2.3 yields the following theorem.

Theorem 5.2. *Let $d \geq 5$ be an integer such that $d \equiv 1 \bmod 4$, k a rational and ε a positive real such that conditions* (10) *and* (11) *are satisfied, where $d' = \lfloor d/4 \rfloor$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least d, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Note that for $d = 5$, the theorem shows that $g(k, 5) \leq 1/k$ for $k \in [2, 3)$.

6 Distances congruent to three mod four

As in the previous sections, we start with showing upper bounds on $g(k, d)$ for $d \equiv 3 \bmod 4$ such that k and d satisfies the condition $2 \leq k < 2 + \dfrac{2}{d-3}$.

Lemma 6.1. *Let ε be a positive real and n, p, q and d positive integers such that $d \equiv 3 \bmod 4$ and $p/q \geq 2$. If the conditions*

$$2 \leq k < 2 + \frac{1}{2d'} \quad \textit{and} \tag{13}$$

$$\varepsilon + \varepsilon k \sum_{j=0}^{d'-1} (k-1)^{2j+1} \geq 1 \tag{14}$$

are satisfied, where $d' = \lfloor d/4 \rfloor$ and $k = p/q$, then any fractional $(k+\varepsilon)$-precoloring of the special vertices of $U^n_{p,q,d}$ can be extended to a fractional $(k+\varepsilon)$-coloring of $U^n_{p,q,d}$.

Proof. For the fourth time, we can limit ourselves to ε that give equality in (14):

$$\varepsilon = \left(1 + k \sum_{j=0}^{d'-1} (k-1)^{2j+1}\right)^{-1}.$$

For later in the proof we observe that these ε trivially satisfy

$$\varepsilon + \varepsilon k \sum_{j=0}^{d'-2} (k-1)^{2j+1} \leq 1. \tag{15}$$

For the universal graph $U^n_{p,q,d}$, let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and f_o be a mapping as described in Proposition 2.5. Analogously to the proof of Lemma 5.1, for each ray R_i we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 2d'+1)$ of the base of R_i is colored by a subset of $f_o(A)$, and the special vertex of R_i is colored by C_i.

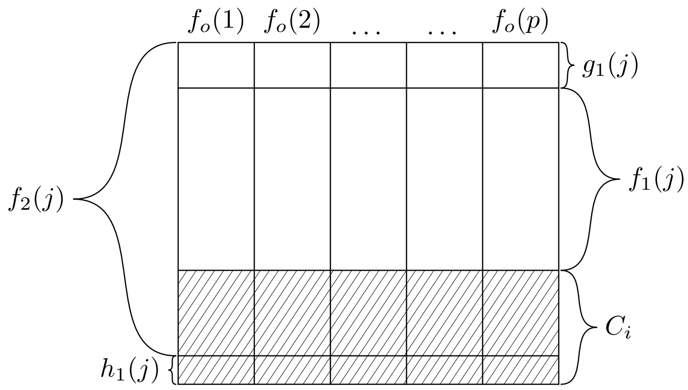

Figure 12: The construction of a fractional coloring in Lemma 6.1 for $d = 7$.

Fix a ray R_i and let s be the special vertex of R_i. For an integer $\ell \in [2d']$, let $V_\ell \subseteq V(R_i)$ be the set of vertices at distance ℓ from s, and let $V_{2d'+1}$ be the set of vertices of R_i at distance at least $2d'+1$ from s. Similarly as in the proof of Lemma 5.1, the vertices of the base of R_i form a subset of $V_{2d'+1}$ and V_ℓ forms an independent set in R_i for $\ell \in [2d']$.

We now construct functions $f_x : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, $g_y : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h_z : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, for $x \in [2d'+1]$, $y \in [d']$ and $z \in [d']$ as follows. For $a \in [p]$ and $j = d'-1, d'-2, \dots, 1$, we sequentially define:

- $h_{d'}(a)$ as an arbitrary subset of $f_o(a) \cap C_i$ of measure $\dfrac{\varepsilon}{p}$, and
- $h_j(a)$ as an arbitrary subset of $(f_o(a) \cap C_i) \setminus \bigcup_{j'=j+1}^{d'} h_{j'}(a)$ of measure $\dfrac{\varepsilon k}{p}(k-1)^{2(d'-j)-1}$.

Next, we sequentially define for $a \in [p]$ and $m = d', d'-1, \dots, 1$

- $g_m(a)$ as an arbitrary subset of $(f_o(a) \setminus C_i) \setminus \bigcup_{m'=m+1}^{d'} g_{m'}(a)$ of measure $\dfrac{\varepsilon k}{p}(k-1)^{2(d'-m)}$,
- $f_{2d'+1}(a) := f_o(a) \setminus h_{d'}(a)$,
- $f_{2m+1}(a) := f_{2m+2}(a) \setminus h_m(a)$ for $m < d'$, and
- $f_{2m}(a) := f_{2m+1}(a) \setminus g_m(a)$.

Finally, we define $f_1(a) := f_2(a) \setminus C_i$ for every $a \in [p]$. Similarly as in the proof of Lemma 5.1, these functions exist if and only if condition (15) is satisfied. The described construction of the functions is sketched in Figure 12.

Let $\ell \in [2d']$ and $v = (A, \ell') \in V_\ell$. If ℓ is even, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=\ell/2}^{d'} h_j([p]);$$

and if ℓ is odd, we set

$$c_i(v) := f_\ell(A) \cup \bigcup_{j=(\ell+1)/2}^{d'} g_j([p]) \,.$$

Also, set $c_i(s) := C_i$. An analysis analogous to that presented in the proof of Lemma 3.1 yields that c_i is a fractional coloring of the ray R_i with the required properties. $\square$

Lemma 6.1 and Proposition 2.3 together provide the following theorem.

Theorem 6.2. *Let d be a positive integer such that $d \equiv 3 \bmod 4$, k a rational and ε a positive real such that conditions* (13) *and* (14) *are satisfied, where $d' = \lfloor d/4 \rfloor$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least d, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

For $d = 7$, the theorem means that $g(k,7) \le \dfrac{1}{k^2-k+1}$ for $k \in [2, 2.5)$. We close this section by showing an upper bound on $g(k,7)$ for $k \in [2.5, 3)$.

Theorem 6.3. *Let k be a positive rational less than 3 and ε a positive real such that $\varepsilon \ge \dfrac{1}{k+1}$. If G is a fractionally k-colorable graph and W is a subset of its vertex set with pairwise distance at least seven, then any fractional $(k+\varepsilon)$-precoloring of W can be extended to a fractional $(k+\varepsilon)$-coloring of G.*

Proof. By Proposition 2.3, it is enough to consider the universal graphs $U^n_{p,q,6}$, where $p/q = k$ and $n \in \mathbb{N}$, and an arbitrary precoloring of its special vertices. Furthermore, we may assume $\varepsilon \le 1$, since $g(k,3) = 1$ for every $k \ge 2$ by Theorem 1.3.

As in the proofs of Lemmas 5.1 and 6.1, let C_i, for $i \in \left[n\binom{p}{q}\right]$, be a precoloring of the special vertices and let f_o be a mapping as described in Proposition 2.5. For each ray R_i we find a fractional coloring c_i that satisfies the following: for every set $A \in \binom{[p]}{q}$, each vertex $v = (A, 3)$ of the base of R_i is colored by a subset of $f_o(A)$, and the special vertex of R_i is colored by C_i.

Fix a ray R_i and let s be the special vertex of R_i. By symmetry, it is enough to consider the case where R_i is a copy of $R^{[q]}_{p,q,3}$. We construct functions $g_2 : [p] \hookrightarrow 2^{[0,k+\varepsilon)}$, $g_1 : [q] \hookrightarrow 2^{[0,k+\varepsilon)}$ and $h : [q] \hookrightarrow 2^{[0,k+\varepsilon)}$ as follows. For $j \in [q]$ and $j' \in [p] \setminus [q]$ we define:

- $g_2(j)$ as an arbitrary subset of $f_o(j) \setminus C_i$ of measure $\dfrac{\varepsilon}{p}$,
- $g_2(j')$ as an arbitrary subset of $f_o(j') \cap C_i$ of measure $\dfrac{\varepsilon}{p}$, and
- $g_1(j)$ as an arbitrary subset of $(f_o(j) \setminus C_i) \setminus g_2(j)$ of measure $\dfrac{\varepsilon}{p}(k-1)$.

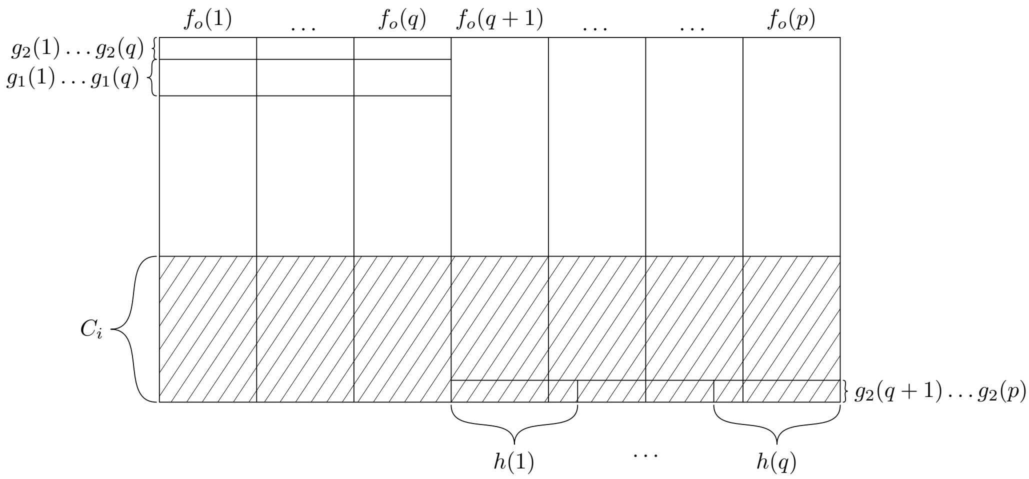

Figure 13: The construction of a fractional coloring in Theorem 6.3.

Note that these functions exist if and only if $\varepsilon \leq 1$. Next, we define sets $h(1), h(2), \dots, h(q)$ as an arbitrary equipartition of $g_2([p] \setminus [q])$ into q parts of measure $\frac{\varepsilon}{p}(k-1)$. The described construction of the functions is sketched in Figure 13.

Recall that the neighborhood of s in R_i forms an independent set. Since we assume that $s = ([q], 0)$, for every neighbor (A, ℓ') of s we have $A \cap [q] = \varnothing$ and $\ell' = 1$. We now construct a fractional coloring of R_i. Let $v = (A, \ell')$ be a vertex of R_i and let ℓ be the distance of v and s in R_i. We define $c_i(v)$ in the following way:

- if $\ell \geq 3$, then $c_i(v) := f_o(A) \setminus g_2(A)$,
- if $\ell = 2$, then $c_i(v) := \big((f_o(A) \setminus g_2(A)) \setminus g_1(A \cap [q])\big) \cup h(A \cap [q])$,
- if $\ell = 1$, then $c_i(v) := (f_o(A) \setminus C_i) \cup g_1([q]) \cup g_2([q])$, and
- $c_i(s) := C_i$.

An analysis analogous to that presented in the proof of Lemma 3.1 yields that we assigned disjoint sets to any two neighbors in R_i, and that any vertex at distance at least two from s got a set of measure one. Furthermore, for every $A \in \binom{[p]\setminus[q]}{q}$ the set $f_o(A) \setminus C_i$ is disjoint from both $g_1([q])$ and $g_2([q])$, and it has measure $1 - (1-\varepsilon)/k$. Since $g_1([q] \cup g_2([q])$ has measure ε and for $\varepsilon \geq 1/(k+1)$ we have $\varepsilon \geq (1-\varepsilon)/k$, it follows that c_i is a fractional coloring with the required properties. □

7 Open problems

Determining further values of $g(k, d)$ seems to require additional knowledge on the structure of independent sets and fractional colorings in Kneser graphs. We believe that our upper bounds presented in Theorems 3.2, 4.2, 5.2, 6.2, and 6.3 are tight. In particular, for distances $d = 5, 6$ and 7, we conjecture the following.

Conjecture 1. *For $k \in [2,3)$ we have* $g(k,5) = \dfrac{1}{k}$.

Conjecture 2. *For $k \in [2,2.5)$ we have* $g(k,6) = \frac{1}{2}\big(\sqrt{k^2+4/(k-1)} - k\big)$.

Conjecture 3. *For $k \in [2,2.5)$ we have* $g(k,7) = \dfrac{1}{k^2-k+1}$, *while for $k \in [2.5,3)$ we have* $g(k,7) = \dfrac{1}{k+1}$.

We also believe that the function $g(k,d)$ is discontinuous for more values of k as d grows. In particular, the following seems as a reasonable expectation.

Conjecture 4. *For a fixed integer $d \geq 4$, the function $g(k,d)$ is discontinuous at $k \in [2,\infty)$ if and only if $k = 2 + 1/m$ with $m \in \{1,2,\ldots,\lfloor d/2 \rfloor - 1\}$.*

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