Universal structures with forbidden homomorphisms

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Abstract

We relate the existence problem of universal objects to the properties of corresponding enriched categories (lifts or expansions). In particular, extending earlier results, we prove that for every regular set $\mathcal{F}$ of finite connected structures there exists a (countable) ω-categorical universal structure $\mathbf{U}$ for the class $\mathrm{Forb_h}(\mathcal{F})$ (of all countable structures not containing any homomorphic image of a member of $\mathcal{F}$). We employ a technique known as homogenization. The universal object $\mathbf{U}$ is the shadow (reduct) of an ultrahomogeneous structure $\mathbf{U}'$.

We also put the results of this paper in the context of homomorphism dualities and constraint satisfaction problems. This leads to an alternative proof of the characterization of finite dualities (given by Tardif and Nešetřil) as well as of the characterization of infinite-finite dualities for classes of relational trees given by P. L. Erdős, Pálvölgyi, Tardif and Tardos.

The notion of regular families of structures is motivated by the recent characterization of infinite-finite dualities for classes of relational forests (itself related to regular languages). We show how the notion of a regular family of relational trees can be extended to regular families of relational structures. This gives a partial characterization of the existence of a (countable) ω-categorical universal object for classes $\mathrm{Forb_h}(\mathcal{F})$.

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1 Introduction

We review first a few well known concepts and facts.

A *relational structure* (or simply *structure*) $\mathbf{A}$ is a pair $(A, (R^i_{\mathbf{A}} : i \in I))$, where $R^i_{\mathbf{A}} \subseteq A^{\delta_i}$ (i.e., $R^i_{\mathbf{A}}$ is a δ_i-ary relation on A). The family $(\delta_i : i \in I)$ is called the *type* Δ. The type is usually fixed and understood from the context. We consider only finite types. If the set A is finite we call $\mathbf{A}$ a *finite structure*. We consider only countable or finite structures. The class of all (countable) relational structures of type Δ will be denoted by $\mathrm{Rel}(\Delta)$. The class $\mathrm{Rel}(\Delta), \Delta = (\delta_i; i \in I)$, is fixed throughout this paper. Unless otherwise stated all structures $\mathbf{A}, \mathbf{B}, \ldots$ belong to $\mathrm{Rel}(\Delta)$.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B} = (B, (R^i_{\mathbf{B}} : i \in I))$ is a mapping $f : A \to B$ such that $(x_1, x_2, \ldots, x_{\delta_i}) \in R^i_{\mathbf{A}}$ implies $(f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R^i_{\mathbf{B}}$, for each $i \in I$. For given structures $\mathbf{A}$ and $\mathbf{B}$ we will denote the existence of homomorphism $f : \mathbf{A} \to \mathbf{B}$ by $\mathbf{A} \to \mathbf{B}$ and the non-existence by $\mathbf{A} \nrightarrow \mathbf{B}$. If f is one-to-one then f is called a *monomorphism*. A monomorphism f such $(x_1, x_2, \ldots, x_{\delta_i}) \in R^i_{\mathbf{A}}$ if and only if $(f(x_1), f(x_2), \ldots, f(x_{\delta_i})) \in R^i_{\mathbf{B}}$ for each $i \in I$ is called an *embedding*.

Given a family of relational structures $\mathcal{F}$, by $\mathrm{Forb_h}(\mathcal{F})$ we denote the class of all relational structures $\mathbf{A}$ for which there is no homomorphism $\mathbf{F} \to \mathbf{A}$, for any $\mathbf{F} \in \mathcal{F}$. Formally,

$$\mathrm{Forb_h}(\mathcal{F}) = \{\mathbf{A}; \forall_{\mathbf{F} \in \mathcal{F}} \mathbf{F} \nrightarrow \mathbf{A}\}.$$

Given a class $\mathcal{K}$ of countable structures, an object $\mathbf{U} \in \mathcal{K}$ is called *universal* for $\mathcal{K}$ if for every object $\mathbf{A} \in \mathcal{K}$ there exists an embedding $\mathbf{A} \to \mathbf{U}$.

For a class $\mathcal{K}$ of countable relational structures, we denote by $\mathrm{Age}(\mathcal{K})$ the class of all finite structures isomorphic to a substructure of some $\mathbf{A} \in \mathcal{K}$ and call it the *age of* $\mathcal{K}$. Similarly, for a relational structure $\mathbf{A}$, the age of $\mathbf{A}$, $\mathrm{Age}(\mathbf{A})$, is $\mathrm{Age}(\{\mathbf{A}\})$.

A structure $\mathbf{A}$ is *ultrahomogeneous* (sometimes also simply called *homogeneous*) if every isomorphism between two induced finite substructures of $\mathbf{A}$ can be extended to an automorphism of $\mathbf{A}$. A structure $\mathbf{G}$ is *generic* for the class $\mathcal{K}$ if it is universal for $\mathcal{K}$ and ultrahomogeneous.

The key property of the age of any ultrahomogeneous structure is described by the following concept. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be relational structures, α an embedding of $\mathbf{C}$ into $\mathbf{A}$, and β an embedding of $\mathbf{C}$ into $\mathbf{B}$. An *amalgamation of* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$ is a triple $(\mathbf{D}, \gamma, \delta)$, where $\mathbf{D}$ is a relational structure, γ an embedding $\mathbf{A} \to \mathbf{D}$ and δ an embedding $\mathbf{B} \to \mathbf{D}$ such that $\gamma \circ \alpha = \delta \circ \beta$. Less formally, an amalgamation "glues together" the structures $\mathbf{A}$ and $\mathbf{B}$ into a single substructure of $\mathbf{D}$ such that copies of $\mathbf{C}$ coincide. See Figure 1.

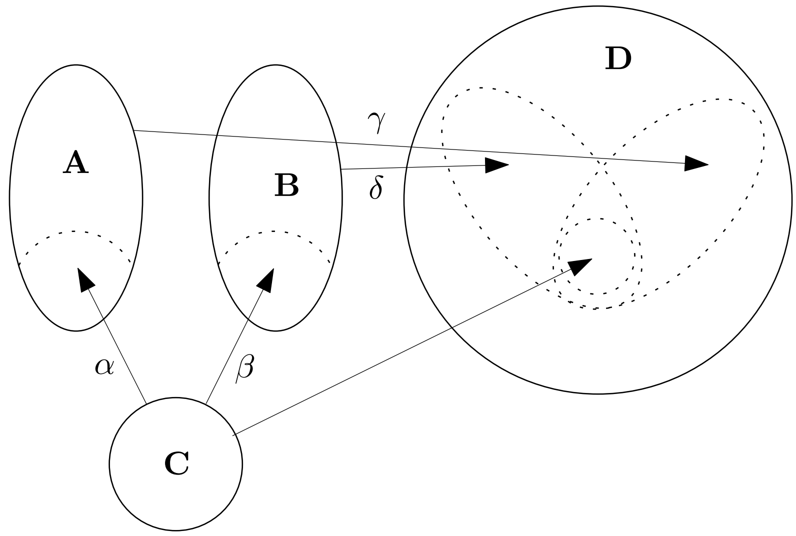

Figure 1: Amalgamation of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$.

Often the vertex sets of structures $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ can be chosen in such a way that the embeddings α and β are identity mappings. In this case, for brevity, we shall call an amalgamation of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$ simply an *amalgamation of* $\mathbf{A}$ *and* $\mathbf{B}$ *over* $\mathbf{C}$. Similarly, for an amalgamation $(\mathbf{D}, \gamma, \delta)$ of a given $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$ we are often interested in the structure $\mathbf{D}$ alone. In this case we shall call the structure $\mathbf{D}$ an amalgamation of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$ (omitting the embeddings γ and δ).

We say that an amalgamation is *strong* when $\gamma(x) = \delta(x')$ if and only if $x \in \alpha(C)$ and $x' \in \beta(C)$. Less formally, a strong amalgamation glues together $\mathbf{A}$ and $\mathbf{B}$ with an overlap no greater than the copy of $\mathbf{C}$ itself. A strong amalgamation is *free* if there are no relations of $\mathbf{D}$ spanning both vertices of $\gamma(A)$ and $\delta(B)$ that are not images of some relations of structure $\mathbf{A}$ or $\mathbf{B}$ via the embedding γ or δ, respectively.

A class $\mathcal{K}$ of finite relational structures is called an *amalgamation class* if the following conditions hold:

1. (*Hereditary property*) For every $\mathbf{A} \in \mathcal{K}$ and induced substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$.

2. (*Amalgamation property*) For $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and α an embedding of $\mathbf{C}$ into $\mathbf{A}$, β an embedding of $\mathbf{C}$ into $\mathbf{B}$, there exists $(\mathbf{D}, \gamma, \delta), \mathbf{D} \in \mathcal{K}$, that is an amalgamation of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \alpha, \beta)$.

3. $\mathcal{K}$ is closed under isomorphism.

4. $\mathcal{K}$ has only countably many mutually non-isomorphic structures. (This is always the case in our setting of finite types).

The following classical result establishes the correspondence between amalgamation classes and ultrahomogeneous structures.

Theorem 1.1 (Fraïssé [10, 12]) *(a) A class $\mathcal{K}$ of finite structures is the age of a countable ultrahomogeneous structure* $\mathbf{G}$ *if and only if* $\mathcal{K}$ *is an amalgamation class.*

(b) If the conditions of (a) are satisfied then the structure $\mathbf{G}$ *is unique up to isomorphism.*

The ultrahomogeneous structure $\mathbf{G}$ such that $\mathrm{Age}(\mathbf{G}) = \mathcal{K}$ is called the *Fraïssé limit* of $\mathcal{K}$. We say that structure $\mathbf{A}$ is *younger* than structure $\mathbf{B}$ if $\mathrm{Age}(\mathbf{A})$ is a subset of $\mathrm{Age}(\mathbf{B})$. Every ultrahomogeneous structure $\mathbf{G}$ has the property that it is universal for the class $\mathcal{K}$ of all countable structures younger than $\mathbf{G}$. It follows that all ultrahomogeneous structures are also universal and generic for the class $\mathcal{K}$.

A countably infinite structure is called *ω-categorical* if all countable models of its first order theory are isomorphic. We use the following characterization of ω-categorical structures given by Engeler [6], Ryll-Nardzewski [20] and Svenonius [21].

Theorem 1.2 *For a countable first order structure* $\mathbf{A}$*, the following conditions are equivalent:*

1. $\mathbf{A}$ *is ω-categorical.*
2. *The automorphism group of* $\mathbf{A}$ *has only finitely many orbits on n-tuples, for every n.*

Lifts and shadows. Let $\Delta' = (\delta'_i; i \in I')$ be a type containing type Δ. (By this we mean $I \subseteq I'$ and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{X} \in \mathrm{Rel}(\Delta')$ may be viewed as a structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}; i \in I)) \in \mathrm{Rel}(\Delta)$ together with some additional relations for $i \in I' \setminus I$. To make this more explicit, these additional relations will be denoted by $X^i_{\mathbf{X}}, i \in I' \setminus I$. Thus a structure $\mathbf{X} \in \mathrm{Rel}(\Delta')$ will be written as

$$\mathbf{X} = (A, (R^i_{\mathbf{A}}; i \in I), (X^i_{\mathbf{X}}; i \in I' \setminus I)),$$

and, abusing notation, more briefly as

$$\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, X^2_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}}).$$

We call $\mathbf{X}$ a *lift* of $\mathbf{A}$ and $\mathbf{A}$ is called the *shadow* of $\mathbf{X}$. In this sense the class $\mathrm{Rel}(\Delta')$ is the class of all lifts of $\mathrm{Rel}(\Delta)$. Conversely, $\mathrm{Rel}(\Delta)$ is the class of all shadows of $\mathrm{Rel}(\Delta')$. If all extended relations are unary, the lift is called *monadic*. In the context of monadic lifts, the *color* of vertex v is the set $\{i; (v) \in X^i_{\mathbf{U}}\}$. Note that a lift is also in the model-theoretic setting called an *expansion* (as we are expanding our relational language) and a shadow a *reduct* (as we are reducing it). (Our terminology is motivated by a computer science context; see [14].) Unless stated explicitely, we shall use letters $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ for shadows (in $\mathrm{Rel}(\Delta)$) and letters $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ for lifts (in $\mathrm{Rel}(\Delta')$).

For a lift $\mathbf{X} = (\mathbf{A}, X^1_{\mathbf{X}}, \ldots, X^N_{\mathbf{X}})$ we denote by $\mathrm{Sh}(\mathbf{X})$ the relational structure $\mathbf{A}$, i.e. its shadow. (Sh is called a *forgetful functor.*) Similarly, for a class $\mathcal{K}'$ of lifted objects, we denote by $\mathrm{Sh}(\mathcal{K}')$ the class of all shadows of structures in $\mathcal{K}'$.

Homogenization. Many naturally defined classes $\mathcal{K}$ of relational structures contain universal structures that are ω-categorical. Because ω-categoricity can be seen as a weaker notion of ultrahomogeneity, it is natural to construct ω-categorical universal structures as shadows of ultrahomogeneous structures. Such construction is called *homogenization.* Covington [5] provided a sufficient condition for the existence of a universal structure for a given class $\mathcal{K}$ that is a shadow of an ultrahomogeneous structure by means of amalgamation failures. This concept in fact relaxes the Fraïssé Theorem.

However, not all universal structures are constructed by means of homogenization. A necessary and sufficient condition for the existence of a universal structure for the class defined by forbidden monomorphisms from a finite family $\mathcal{F}$ of connected graphs was given by Cherlin, Shelah and Shi [3]. Here the classes are characterized by means of local finiteness of the algebraic closure operator. The techniques of [3] are motivated by proofs of the non-existence of a universal structure for a given class. The universal structure is not constructed by an explicit amalgamation argument.

Our motivation and results. Our motivation stems from several sources. First, we seek a more streamlined and combinatorial proof of the following corollary of the aforementioned result of Cherlin, Shelah and Shi.

Theorem 1.3 ([3]) *For every finite family $\mathcal{F}$ of finite connected graphs there is an ω-categorical universal graph for the class* $\mathrm{Forb_h}(\mathcal{F})$.

We prove a stronger form of Theorem 1.3 by an explicit amalgamation argument. A similar construction can be also be obtained by characterizing the amalgamation failures and applying Covington's homogenization method. Our lifts are, however, different and, for the first time, we avoid using the model-theoretic concept of existential completeness.

We are interested in the structure of lifts constructed for a given (possibly infinite) family $\mathcal{F}$. In special cases we relate lifts to the concept of homomorphism dualities. Motivated by a recent characterization of infinite-finite dualities by P. L. Erdős, Pálvölgyi, Tardif, Tardos [7], we introduce a notion of regular families of relational structures. These (possibly infinite) families of structures generalize regular forests, used in [7] to characterize infinite-finite dualities.

In Section 3 we strengthen Theorem 1.3 by proving the existence of a universal structure for $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F})$, where $\mathcal{F}$ is a regular family of finite connected structures.

In Section 4 we show the non-existence of an ω-categorical universal structure for $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F})$ for certain non-regular families $\mathcal{F}$, and give a partial characterization of such families.

Finally, in Section 5 we relate our results to homomorphism dualities and constraint satisfaction problems. We show that for the classes $\mathcal{F}$ consisting of regular relational trees the universal structure has a finite retract. This gives an alternative construction of graph duals and also an alternative proof of the characterization of homomorphism dualities.

2 Regular families of structures and $\mathcal{F}$-lifts

Let $\mathcal{F}$ be a fixed set of finite connected relational structures. For the construction of a universal structure of $\mathrm{Forb}_{\mathrm{h}}(\mathcal{F})$ we use special lifts, called $\mathcal{F}$-lifts. The definition of an $\mathcal{F}$-lift is easy and resembles decomposition techniques standard in graph theory, and thus we adopt a similar terminology. First we overview some elementary graph-theoretic notions, see [15, 2] for details.

For a structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}, i \in I))$, the *Gaifman graph* (in combinatorics often called *2-section*) is the graph $G_{\mathbf{A}}$ with vertices A and all those edges which are a subset of a tuple of a relation of $\mathbf{A}$, i.e., $G = (A, E)$, where $x, y \in E$ if and only if $x \neq y$ and there exists a tuple $\vec{v} \in R^i_{\mathbf{A}}$, $i \in I$, such that $x, y \in \vec{v}$.

We adopt the following standard graph-theoretic notions for relational structures. We call structure $\mathbf{A}$ *connected* if its Gaifman graph $G_{\mathbf{A}}$ is connected. For a structure $\mathbf{A}$ and subset of its vertices $B \subseteq A$, we denote by $N_{\mathbf{A}}(B)$ the *neighborhood* of the set B, that is all vertices of $A \setminus B$ connected in the Gaifman graph $G_{\mathbf{A}}$ by an edge to a vertex of B. We denote by $\mathbf{A} \setminus B$ the structure induced on $A \setminus B$ by $\mathbf{A}$. Similarly we denote by $G_{\mathbf{A}} \setminus B$ the graph created from the Gaifman graph $G_{\mathbf{A}}$ by removing vertices B.

A *g-cut* in $\mathbf{A}$ is a subset C of A such that the Gaifman graph $G_{\mathbf{A}}$ is disconnected by removing set C. That is, there are vertices $u, v \in A \setminus C$ that belong to the same connected component of $G_{\mathbf{A}}$ but to different connected components of $G_{\mathbf{A}} \setminus C$. A *cut* in $\mathbf{A}$ is subset C of A such that there are vertices $u, v \in A \setminus C$ that belong to the same connected component of $\mathbf{A}$ but to different connected components of $\mathbf{A} \setminus C$.

Observe that not every cut is a g-cut. With relations of arity greater than 2, $G_{\mathbf{A} \setminus C}$ may be different from $G_{\mathbf{A}} \setminus C$.

For g-cut C in relational structure $\mathbf{A}$ a structure $\mathbf{A}_1$ is a *g-component* of $\mathbf{A}$ with g-cut C if $\mathbf{A}_1$ is induced by $\mathbf{A}$ on some connected component of $G_{\mathbf{A}} \setminus C$.

We will make use of the following simple observation about the neighborhood and g-components.

Observation 2.1 *Let $\mathbf{A_1}$ be a g-component of $\mathbf{A}$ with g-cut C. Then the neighborhood $N_{\mathbf{A}}(A_1)$ is a subset of C. Moreover $N_{\mathbf{A}}(A_1)$ is a g-cut and $\mathbf{A}_1$ is one of the g-components of $\mathbf{A}$ with g-cut $N_{\mathbf{A}}(A_1)$.*

Given a structure $\mathbf{A}$ with g-cut C and two (induced) substructures $\mathbf{A}_1$ and $\mathbf{A}_2$, we say that C *g-separates* $\mathbf{A}_1$ and $\mathbf{A}_2$ if there are g-components $\mathbf{A}'_1 \neq \mathbf{A}'_2$ of $\mathbf{A}$ with a g-cut C such that $A_1 \subseteq A'_1$ and $A_2 \subseteq A'_2$.

Definition 2.1 *Let C be a g-cut in structure $\mathbf{A}$. Let $\mathbf{A}_1 \neq \mathbf{A}_2$ be two g-components of $\mathbf{A}$ with g-cut C. We call C* minimal g-separating g-cut *for $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$ if $C = N_{\mathbf{A}}(A_1) = N_{\mathbf{A}}(A_2)$.*

For brevity, we can omit one or both g-components when speaking about a minimal g-separating g-cut. Explicitely, we call g-cut C *minimal g-separating* for $\mathbf{A}_1$ in $\mathbf{A}$ if there exists another structure $\mathbf{B}$ such that C is minimal g-separating for $\mathbf{A}_1$ and $\mathbf{B}$ in $\mathbf{A}$. A g-cut C is *minimal g-separating* in $\mathbf{A}$ if there exists structures $\mathbf{B}_1$ and $\mathbf{B}_2$ such that C is minimal g-separating for $\mathbf{B}_1$ and $\mathbf{B}_2$ in $\mathbf{A}$.

The name of minimal g-separating g-cut is justified by the following (probably folkloristic) proposition.

Proposition 2.1 *Let $\mathbf{A}$ be a connected relational structure, C a g-cut in $\mathbf{A}$ and $\mathbf{A}_1$ and $\mathbf{A}_2$ (induced) substructures of $\mathbf{A}$ g-separated by C. Then there exists a minimal g-separating g-cut $C' \subseteq C$ that g-separates $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$. Moreover if $N_{\mathbf{A}}(A_1) \subseteq C$ (or, equivalently, $\mathbf{A}_1$ is a g-component of $\mathbf{A}$ with g-cut C), then $C' \subseteq N_{\mathbf{A}}(A_1)$.*

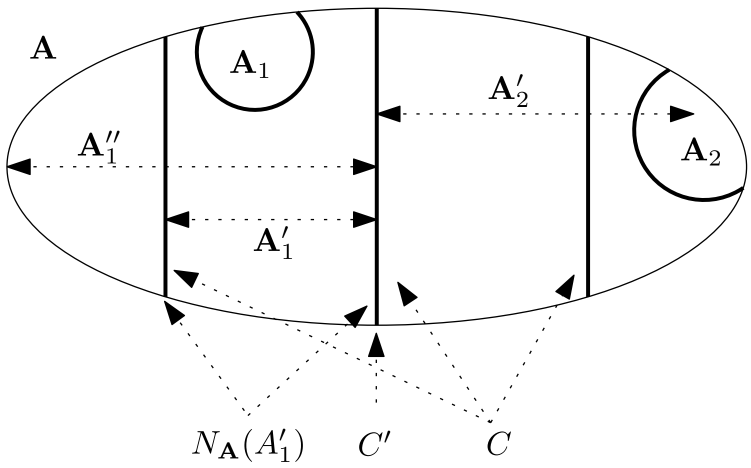

Figure 2: Construction of a minimal g-separating g-cut C' g-separating $\mathbf{A}_1$ and $\mathbf{A}_2$ in $\mathbf{A}$.

Proof. We will construct a series of g-cuts and g-components as depicted in Figure 2.

Denote by $\mathbf{A}'_1$ the g-component of $\mathbf{A}$ with g-cut C containing $\mathbf{A}_1$ (and thus not containing $\mathbf{A}_2$). By Observation 2.1, $N_{\mathbf{A}}(A'_1) \subseteq C$ is g-cut that g-separates $\mathbf{A}'_1$ and $\mathbf{A}_2$ (because $\mathbf{A}'_1$ is also g-component of $\mathbf{A}$ with g-cut $N_{\mathbf{A}}(A'_1)$ and $\mathbf{A}'_1$ do not contain $\mathbf{A}_2$).

Now consider g-component $\mathbf{A}'_2$ of $\mathbf{A}$ with g-cut $N_{\mathbf{A}}(A'_1)$ containing $\mathbf{A}_2$. Put $C' = N_{\mathbf{A}}(A'_2)$. By Observation 2.1, $C' \subseteq N_{\mathbf{A}}(A'_1) \subseteq C$ is g-cut and $\mathbf{A}'_2$ (not containing $\mathbf{A}_1$) is one of its g-components.

Denote by $\mathbf{A}''_1$ the g-component of $\mathbf{A}$ with g-cut C' containing $\mathbf{A}_1$. It follows that C' g-separates $\mathbf{A}''_1$ (that contains $\mathbf{A}_1$) and $\mathbf{A}'_2$ (that contains $\mathbf{A}_2$).

To see that C' is minimal g-separating for $\mathbf{A}''_1$ and $\mathbf{A}'_2$ it remains to show that every vertex in $C' = N_{\mathbf{A}}(A'_2)$ is also in $N_{\mathbf{A}}(A''_1)$. This is true because every vertex of C' is in $N_{\mathbf{A}}(A'_1)$ and $\mathbf{A}'_1$ is substructure of $\mathbf{A}''_1$.

□

Observe that every inclusion minimal g-cut is also minimal g-separating, but not vice versa. Every minimal g-separating g-cut $C' \subset C$ that g-separates $\mathbf{A}_1$ and $\mathbf{A}_2$ is however also inclusion minimal g-cut that separates $\mathbf{A}_1$ and $\mathbf{A}_2$.

If C is a set of vertices then $\overrightarrow{C}$ will denote a tuple (of length $|C|$) of all the elements of C. Alternatively, $\overrightarrow{C}$ is an arbitrary linear ordering of C. A *rooted structure* $\mathcal{P}$ is a pair $(\mathbf{P}, \overrightarrow{R})$ where $\mathbf{P}$ is a relational structure and $\overrightarrow{R}$ is an tuple consisting of distinct vertices of $\mathbf{P}$. $\overrightarrow{R}$ is called the *root* of $\mathcal{P}$ and the size of $\overrightarrow{R}$ is the *arity* of $\mathcal{P}$. We say that rooted structures $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ and $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ are *isomorphic* if there is a function $f : P_1 \to P_2$ that is an

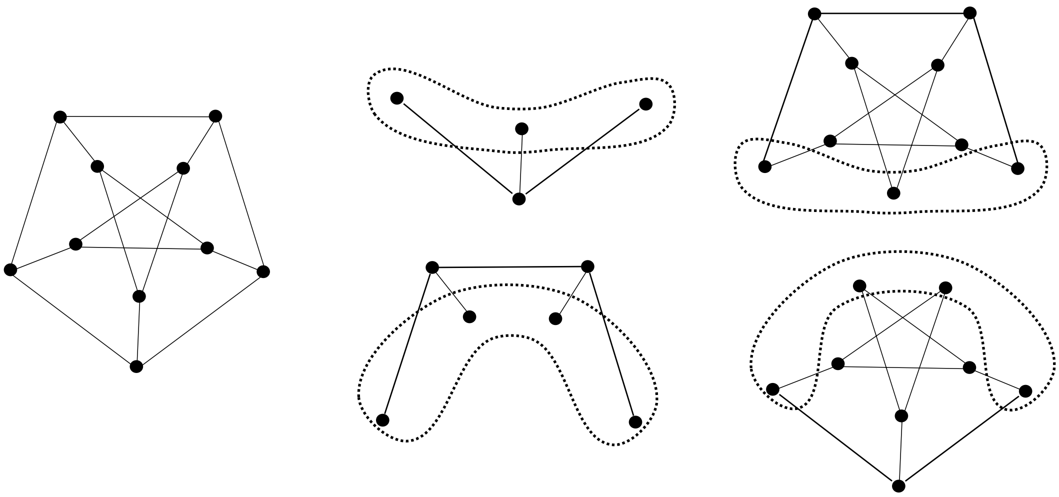

Figure 3: Pieces of the Petersen graph up to isomorphism (and permutations of roots).

isomorphism of structures $\mathbf{P}_1$ and $\mathbf{P}_2$ and f restricted to $\overrightarrow{R}_1$ is a monotone bijection between $\overrightarrow{R}_1$ and $\overrightarrow{R}_2$ (we denote this $f(\overrightarrow{R}_1) = \overrightarrow{R}_2$).

The following is the principal notion of this paper:

Definition 2.2 *Let* $\mathbf{A}$ *be a connected relational structure and* R *a minimal g-separating g-cut for structure* $\mathbf{P}$ *in* $\mathbf{A}$*. A* piece *of a relational structure* $\mathbf{A}$ *is then a rooted structure* $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$*, where the tuple* $\overrightarrow{R}$ *consists of the vertices of the g-cut* R *in a (fixed) linear order.*

Note that since $\mathbf{P}$ is the union of a g-component and its neighborhood it follows that the pieces of a connected structure are always connected structures. As an example, pieces of the Petersen graph are shown in Figure 3.

Given rooted structures $(\mathbf{P}, \overrightarrow{R})$ and $(\mathbf{P}', \overrightarrow{R}')$ such that $|R| = |R'|$, denote by $(\mathbf{P}, \overrightarrow{R}) \oplus (\mathbf{P}', \overrightarrow{R}')$ the (possibly rooted) structure created as a free amalgam of $\mathbf{P}$ and $\mathbf{P}'$ with corresponding roots being identified (in the order of $\overrightarrow{R}$ and $\overrightarrow{R}'$). Note that $(\mathbf{P}, \overrightarrow{R}) \oplus (\mathbf{P}', \overrightarrow{R}')$ is defined only if the rooted structure induced by $\mathbf{P}$ on $\overrightarrow{R}$ is isomorphic to the rooted structure induced by $\mathbf{P}'$ on $\overrightarrow{R}'$.

A piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ is *incompatible* with a rooted structure $\mathcal{A}$ if $\mathcal{P} \oplus \mathcal{A}$ is defined and there exists $\mathbf{F} \in F$ that is isomorphic to $\mathcal{P} \oplus \mathcal{A}$. (In the other words, there exists $\mathbf{F}'$ isomorphic to some $\mathbf{F}'' \in \mathcal{F}$, such that $\mathcal{P}$ is piece of $\mathbf{F}'$ and $\mathcal{A}$ is a structure induced on $F' \setminus (P \setminus R)$ by $\mathbf{F}'$ rooted by $\overrightarrow{R}$.)

Assign to each piece $\mathcal{P}$ a set $\mathcal{I}_{\mathcal{P}}$ containing all rooted structures that are incompatible with $\mathcal{P}$. For two pieces $\mathcal{P}_1$ and $\mathcal{P}_2$ put $\mathcal{P}_1 \sim \mathcal{P}_2$ if and only if

$\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2}$. Observe that every equivalence class of $\sim$ contains pieces of the same arity n. We also call n the *arity* of the equivalence class of $\sim$.

Definition 2.3 *A family of finite structures $\mathcal{F}$ is called* regular *if there are only finitely many equivalence classes of $\sim$ on the family of all pieces of $\mathcal{F}$.*

The notion of regular family is a generalization of that of a regular family of forests, introduced in [7]. (The term used in [7] was motivated by the connection to regular languages we explain in the following examples.)

Example. All finite families $\mathcal{F}$ of finite structures are regular. Examples of infinite families $\mathcal{F}$ include the following:

1. The family $\mathcal{F}_{\mathrm{odd}}$ consisting of all graph cycles of odd length. All pieces of $\mathcal{F}_{\mathrm{odd}}$ are paths rooted by initial vertex and terminal vertex. There are only two equivalence classes of the pieces: paths of odd length and paths of even length.

2. The family $\mathcal{F}_{\mathrm{oriented}}$ consisting of those orientations of graph cycles where all edges are oriented in the same direction. Pieces of $\mathcal{F}_{\mathrm{oriented}}$ are oriented paths with all edges in a forward direction with roots on initial and terminal vertex. Consequently there are only two equivalence classes of pieces: paths with first root on initial vertex and second root on terminal vertex, and paths with first root on terminal vertex and second root on initial vertex.

3. Oriented paths can be described by words on alphabet $\{\leftarrow, \rightarrow\}$. It follows that every language of words on this alphabet corresponds to a family of oriented paths. It is not difficult to show that all regular languages correspond to a regular family of paths. Consequently regular families may have a rich structure; see [7].

   Consider for example the family created by words of the form $\rightarrow\rightarrow (\rightarrow\leftarrow\rightarrow)^n \rightarrow\rightarrow$, $n \geq 1$, where $(\rightarrow\leftarrow\rightarrow)^n$ stands for n repetitions of $\rightarrow\leftarrow\rightarrow$. All these paths are cores and form an antichain. Several other examples of regular families of directed graphs are discussed in [8].

We continue our construction with the following:

Definition 2.4 *We denote by $E_1, \ldots E_N$ the equivalence classes of $\sim$ corresponding to pieces of structures in $\mathcal{F}$. Put $I' = \{1, 2, \ldots, N\}$. The relational structure $\mathbf{X} = (\mathbf{A}, (X^i_{\mathbf{X}} : i \in I'))$ is called the $\mathcal{F}$-*lift *of the relational structure $\mathbf{A}$ when the arities of relations $X^i_{\mathbf{X}}, i \in I'$, correspond to the arity of E_i.*

For a relational structure $\mathbf{A}$*, we define the* canonical lift

$$L(\mathbf{A}) = (\mathbf{A}, X^1_{L(\mathbf{A})}, X^2_{L(\mathbf{A})}, \ldots, X^N_{L(\mathbf{A})})$$

by putting $(v_1, v_2, \ldots, v_l) \in X^i_{L(\mathbf{A})}$ *if and only if there is a piece* $\mathcal{P} = (\mathbf{P}, \overrightarrow{R}) \in E_i$ *such that there is a homomorphism* $f : \mathbf{P} \to \mathbf{A}$ *with* $f(\overrightarrow{R}) = (v_1, v_2, \ldots, v_l)$*.*

Example. As an introduction we provide an explicit description of some lifts of the regular families discussed above.

1. For the family $\mathcal{F}_{\mathrm{odd}}$ there are two new binary relations. In a canonical lift $L(\mathbf{A})$ there is $(u, v) \in X^1_{L(\mathbf{A})}$ if and only if there is a walk of odd length between vertices u and v and $(u, v) \in X^2_{L(\mathbf{A})}$ if and only if there is a walk of even length between u and v.

   For $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F}_{\mathrm{odd}})$ there is no (u, v) such that $(u, v) \in X^1_{L(\mathbf{A})}$ and $(u, v) \in X^2_{L(\mathbf{A})}$. This means that odd cycles can be recognized by the existence of both a walk of even length and a walk of odd length in between a given pair of vertices.

2. For the family $\mathcal{F}_{\mathrm{oriented}}$ there are two new binary relations. In a canonical lift $L(\mathbf{A})$ there is $(u, v) \in X^1_{L(\mathbf{A})}$ if and only if there is an oriented walk from u to v and $(u, v) \in X^2_{L(\mathbf{A})}$ if and only if there is an oriented walk from v to u.

   For $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F}_{\mathrm{oriented}})$ there is no (u, v) such that both $(u, v) \in X^1_{L(\mathbf{A})}$ and $(v, u) \in X^1_{L(\mathbf{A})}$. The same holds for $X^2_{L(\mathbf{A})}$ and in fact the second relation is fully redundant in our construction and can be ignored.

3 Construction of the universal structure

Theorem 3.1 *Let* $\mathcal{F}$ *be a regular family of finite connected relational structures (of a finite type). Then there exists an ultrahomogeneous lift* $\mathbf{U}'$ *with only finitely many new relations such that its shadow* $\mathrm{Sh}(\mathbf{U}')$ *is a universal structure for the class* $\mathrm{Forb_h}(\mathcal{F})$*.*

Moreover, the lift $\mathbf{U}'$ *can be constructed in the following way. Denote by* $\mathcal{L}$ *the class of all induced substructures (sublifts) of canonical lifts* $L(\mathbf{A})$*,* $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$*. Then* $\mathrm{Age}(\mathcal{L})$ *is a amalgamation class (closed for strong amalgams whose shadows are free amalgams) and* $\mathbf{U}'$ *is the Fraïssé limit of* $\mathrm{Age}(\mathcal{L})$*.*

Denote by n *the maximal size of a minimal g-separating g-cut in a structure in* $\mathcal{F}$ *(by regularity of* $\mathcal{F}$ *the size of g-cuts is bounded). Then the arity of extended relations is bounded by* n*.*

This theorem will be proved in the rest of this section. We take time for a simple Lemma.

Given a piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ of structure $\mathbf{F}$, we call $\mathcal{P}' = (\mathbf{P}', \overrightarrow{R}')$ a *subpiece* of $\mathcal{P}$ if $\mathcal{P}'$ is piece of $\mathbf{F}$, $P' \subset P$. We show that a subpiece can be freely replaced by an equivalent subpiece without changing the equivalence class of a given piece.

Lemma 3.2 *Let $\mathcal{P}_1 = (\mathbf{P}_1, \overrightarrow{R}_1)$ be a piece of structure $\mathbf{F}_1 \in \mathcal{F}$, $\mathcal{P}'_1 = (\mathbf{P}'_1, \overrightarrow{R}'_1)$ be a subpiece of $\mathcal{P}_1$ and $\mathcal{P}'_2 = (\mathbf{P}'_2, \overrightarrow{R}'_2)$ a piece such that $\mathcal{P}'_1 \sim \mathcal{P}'_2$.*

Create $\mathcal{P}_2 = (\mathbf{P}_2, \overrightarrow{R}_2)$ as a copy of $\mathcal{P}_1$ with $\mathbf{P}'_1$ replaced by $\mathbf{P}'_2$ identifying $\overrightarrow{R}'_1$ with $\overrightarrow{R}'_2$. Then $\mathcal{P}_2$ is an isomorphic copy of a piece of some $\mathbf{F}_2 \in \mathcal{F}$, and moreover $\mathcal{P}_1 \sim \mathcal{P}_2$.

Proof. Consider some $\mathcal{A} \in \mathcal{I}_{\mathcal{P}_1}$. By definition $\mathcal{P}_1 \oplus \mathcal{A}$ is isomorphic to some structure $\mathbf{F} \in \mathcal{F}$. Let $\mathcal{A}'$ be a rooted structure such that $\mathcal{P}'_1 \oplus \mathcal{A}' = \mathbf{F}$. Because $\mathcal{A}' \in \mathcal{I}_{\mathcal{P}'_2} = \mathcal{I}_{\mathcal{P}'_1}$, we also know that $\mathcal{P}_2 \oplus \mathcal{A} = \mathcal{P}'_2 \oplus \mathcal{A}'$ is isomorphic to some structure in $\mathcal{F}$. We thus have $\mathcal{I}_{\mathcal{P}_1} \subseteq \mathcal{I}_{\mathcal{P}_2}$. By symmetry we also have $\mathcal{I}_{\mathcal{P}_2} \subseteq \mathcal{I}_{\mathcal{P}_1}$. □

For $\mathbf{X} \in \mathcal{L}$ we denote by $W(\mathbf{X})$ one of the structures $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$ such that the structure $\mathbf{X}$ is induced on X by $L(\mathbf{A})$. $W(\mathbf{X})$ is called a *witness* of the fact that $\mathbf{X}$ belongs to $\mathcal{L}$. Note that in this definition, the witness of a finite lift may be infinite structure, because $\mathrm{Forb_h}(\mathcal{F})$ contains infinite structures.

Proof of Theorem 3.1. Clearly it suffices to prove the second part of the theorem. By definition the class $\mathrm{Age}(\mathcal{L})$ is hereditary, isomorphism-closed, and has the joint embedding property. Assuming that $\mathrm{Age}(\mathcal{L})$ has the amalgamation property (with restrictions described), the rest of the theorem follows from the Fraïssé Theorem and the fact that $\mathcal{L}$ is the class of all lifts younger than the Fraïssé limit $\mathbf{U}'$ of $\mathrm{Age}(\mathcal{L})$ and thus $\mathbf{U}'$ is generic for $\mathcal{L}$.

We show the amalgamation property. Consider $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathrm{Age}(\mathcal{L})$. Assume that structure $\mathbf{Z}$ is a substructure induced by both $\mathbf{X}$ and $\mathbf{Y}$ on Z and without loss of generality assume that $X \cap Y = Z$.

Put

$$\mathbf{A} = W(\mathbf{X}),$$

$$\mathbf{B} = W(\mathbf{Y}),$$

$$\mathbf{C} = \mathrm{Sh}(\mathbf{Z}).$$

Because $\mathrm{Age}(\mathcal{L})$ is closed under isomorphism, we can assume that $\mathbf{A}$ and $\mathbf{B}$ are vertex-disjoint with the exception of vertices of $\mathbf{C}$.

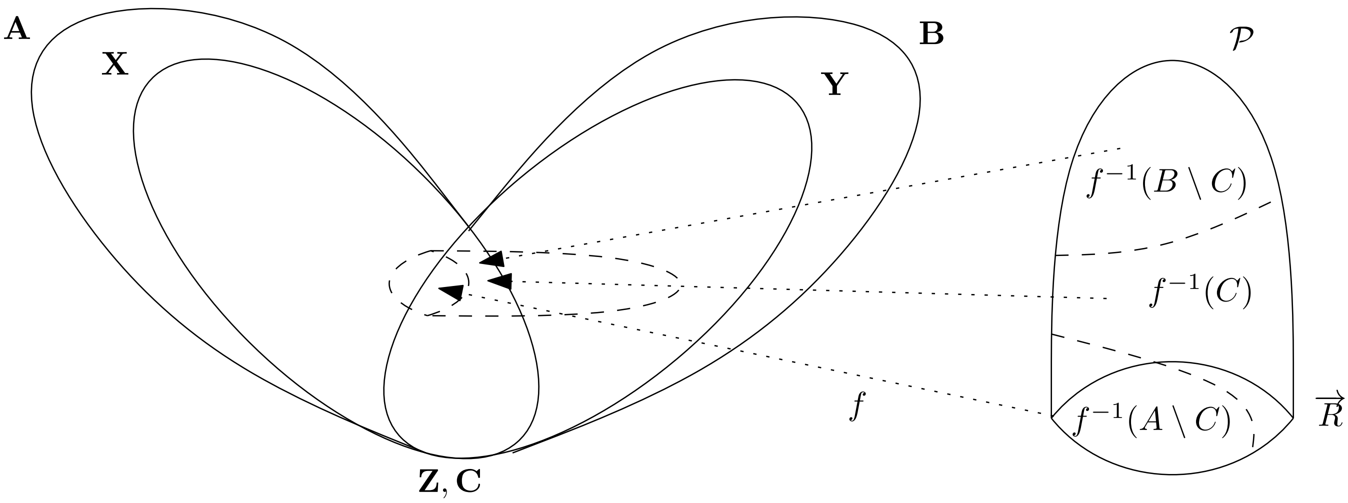

Figure 4: Construction of an amalgamation.

Let $\mathbf{D}$ be the free amalgamation of $\mathbf{A}$ and $\mathbf{B}$ over vertices of $\mathbf{C}$: the vertices of $\mathbf{D}$ are $A \cup B$ and there is $\vec{v} \in R^i_{\mathbf{D}}$ if and only if $\vec{v} \in R^i_{\mathbf{A}}$ or $\vec{v} \in R^i_{\mathbf{B}}$.

We claim that the structure

$$\mathbf{V} = L(\mathbf{D})$$

is a strong amalgamation of $L(\mathbf{A})$ and $L(\mathbf{B})$ over $\mathbf{Z}$ and thus also an amalgamation of $\mathbf{X}, \mathbf{Y}$ over $\mathbf{Z}$. The situation is depicted in Figure 4.

First we show that the substructure induced by $\mathbf{V}$ on A is $L(\mathbf{A})$ and that the substructure induced by $\mathbf{V}$ on B is $L(\mathbf{B})$. In the other words, no new tuples to $L(\mathbf{A})$ or $L(\mathbf{B})$ (and thus none to $\mathbf{X}$ or $\mathbf{Y}$ either) have been introduced. Assume to the contrary that there is a new tuple $(v_1, \ldots, v_t) \in X^k_{\mathbf{V}}$. By symmetry we can assume that $\{v_1, \ldots, v_t\} \subseteq A$. Explicitly, we assume that there is a piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R}) \in E_k$ and a homomorphism f from $\mathbf{P}$ to $\mathbf{D}$ such that $f(\overrightarrow{R}) = (v_1, v_2, \ldots, v_t) \notin X^k_{L(\mathbf{A})}$.

The set of vertices of $\mathbf{P}$ mapped to $L(\mathbf{A})$, $f^{-1}(A)$, is nonempty, because it contains all vertices of $\overrightarrow{R}$. The set $f^{-1}(B \setminus C)$ is nonempty f is not homomorphism from $\mathbf{P}$ to $\mathbf{A}$ (otherwise we would have $(v_1, v_2, \ldots, v_t) \in X^k_{L(\mathbf{A})}$). Because there are no tuples spanning both vertices $A \setminus C$ and vertices $B \setminus C$ in $\mathbf{D}$, and because pieces are connected, we also have $f^{-1}(C)$ nonempty.

We will reason about the decomposition of $\mathcal{P}$ given by $f^{-1}(C)$ and create subpieces containing vertices of $f^{-1}(B \setminus C)$. This requires some careful analysis. The process is depicted in Figure 5. Denote by $\mathbf{F} \in \mathcal{F}$ the structure such that $\mathcal{P}$ is piece of $\mathbf{F}$. The vertices of $f^{-1}(C)$ form a g-cut in $\mathbf{F}$ g-separating any vertex in $f^{-1}(B \setminus C)$ from any vertex in $f^{-1}(A \setminus C)$ (if such a vertex exists) as well as any vertex of $\mathbf{F} \setminus P$.

We further strengthen our assumption on the choice of counter-example

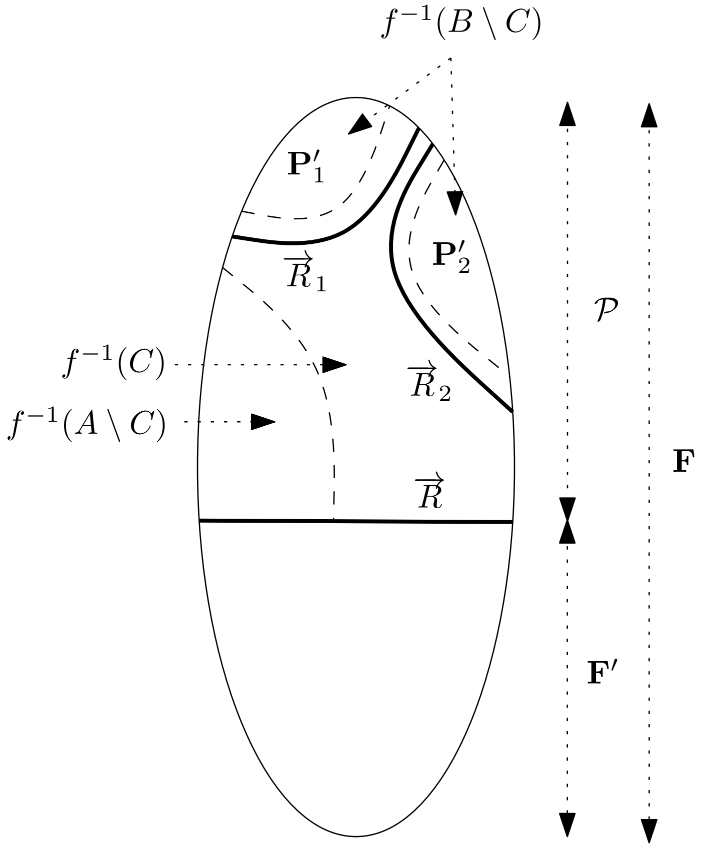

Figure 5: The decomposition of piece $\mathcal{P}$.

(consisting of the choice of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{A}$, $\mathbf{B}$, the piece $\mathcal{P}$ and the homomorphism f):

(a) No subpiece $\mathcal{P}' = (\mathbf{P}', \overrightarrow{R}')$ of $\mathcal{P}$ is a counter-example with homomorphism f. More precisely if $f(R') \subseteq A$, then $f(\overrightarrow{R}') \in X^i_{L(\mathbf{A})}$ for i such that $\mathcal{P}' \in E_i$. Similarly if $f(R') \subseteq B$, then $f(\overrightarrow{R}') \in X^i_{L(\mathbf{B})}$ for i such that $\mathcal{P}' \in E_i$.

(b) If there is some g-component $\mathbf{P}'$ of $\mathbf{F}$ with g-cut $f^{-1}(C)$ contained in $f^{-1}(B \setminus C)$ such that $R \subseteq N_{\mathbf{P}}(P')$, then there exists a g-component $\mathbf{P}''$ of $\mathbf{F}$ with g-cut $f^{-1}(C)$ contained in $f^{-1}(A \setminus C)$ such that $R \subseteq N_{\mathbf{P}}(P'')$.

It easily follows that the existence of counter-example implies the existence of counter-example satisfying (a) and (b). (a) can be made to hold by considering the smallest subpiece of $\mathcal{P}$ that is still a counter-example. If (b) fails we can exchange $\mathbf{A}$ and $\mathbf{B}$ as well as exchange $\mathbf{X}$ and $\mathbf{Y}$. This is possible because $f^{-1}(C)$ g-separates any vertex in $f^{-1}(A \setminus C)$ and any vertex in $f^{-1}(B \setminus C)$ and thus the existence of $\mathbf{P}'$ implies $R \subseteq f^{-1}(C)$.

Denote by $\mathbf{F}'$ the substructure of $\mathbf{F}$ so that R is a minimal g-separating g-cut of $\mathbf{F}'$ and $\mathbf{P} \setminus R$ in $\mathbf{F}$. Denote by $\mathbf{P}'_1, \mathbf{P}'_2, \ldots, \mathbf{P}'_l$ the substructures induced on $\mathbf{P}$ by all connected components of $G_{\mathbf{P}} \setminus f^{-1}(A)$. We aim to find, for every $i = 1, 2, \dots l$, a minimal g-separating g-cut $R_i \subseteq f^{-1}(C)$ of $\mathbf{F}$ that separates $\mathbf{P}'_i$ from $\mathbf{F}'$ and moreover $R_i \not\subseteq R$. This implies the existence of $\mathcal{P}_i = (\mathbf{P}_i, \overrightarrow{R}_i)$ that is a subpiece of $\mathcal{P}$ containing $\mathbf{P}'_i$.

We consider two cases:

1. $R \not\subseteq N_{\mathbf{P}}(P'_i)$: Construct $R_i \subseteq N_{\mathbf{P}}(P'_i)$ as a minimal g-separating g-cut that g-separates $\mathbf{F}'$ and $\mathbf{P}'_i$ in $\mathbf{F}$ (given by Proposition 2 for structure $\mathbf{F}$ and g-cut $f^{-1}(C)$).

   Since pieces are connected and R is a minimal g-separating g-cut in $\mathbf{F}$ for $\mathbf{P} \setminus R$ and $\mathbf{F}'$, we know that $\overrightarrow{R}_i$ must contain some vertex $v \notin R$.

2. $R \subseteq N_{\mathbf{P}}(P'_i)$: In this case consider structure $\mathbf{P}''$ given by (b). Construct R_i as a minimal g-separating g-cut that g-separates $\mathbf{P}''$ and $\mathbf{P}'_i$ in $\mathbf{F}$.

   We show that $R_i \supset R$. Because every vertex $v \in R$ is connected in $G_{\mathbf{P}}$ to a vertex in P_i and a vertex in P'', we have $R_i \supseteq R$. Moreover because pieces with roots removed are g-components, R does not g-separate $\mathbf{P}''$ and $\mathbf{P}'_i$ and thus $R_i \supset R$.

   Since $R_i \supset R$, R_i g-separates $\mathbf{P}'_i$ and $\mathbf{F}'$ in $\mathbf{F}$.

We have constructed a family of subpieces $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_l$ such that $\mathbf{P}'_i$ is contained in $\mathbf{P}_i \setminus \overrightarrow{R}$. It is possible that $\mathcal{P}_i$ is a subpiece of $\mathcal{P}_j$ for some $i \neq j$. Without loss of generality assume that $\mathcal{P}_1, \mathcal{P}_2, \dots \mathcal{P}_{l'}$ is the maximal subset of pieces $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_l$ such that no piece is a subpiece of any other. Obviously $P \setminus f^{-1}(A) = \cup_{i=1,2,\dots l} P'_i$ is a subset of $\cup_{i=1,2,\dots l'} (P_i \setminus R_i)$.

Let e_i be the index of the equivalence class of $\sim$ such that $\mathcal{P}_i \in E_{e_i}$. Now we use assumption (a). All the pieces $\mathcal{P}_i$, $i = 1, \dots, l'$ are subpieces of $\mathcal{P}$. Thus we have that $f(\overrightarrow{R}_i) \in X^{e_i}_{L(\mathbf{D})} \implies f(\overrightarrow{R}_i) \in X^{e_i}_{L(\mathbf{A})}$. Thus there exists a piece $\mathcal{P}^A_i = (\mathbf{P}^A_i, \overrightarrow{R}^A_i)$, $\mathcal{P}^A_i \sim \mathcal{P}_i$, and a homomorphism f^A_i from $\mathbf{P}^A_i$ to $\mathbf{A}$ such that $f^A_i(\overrightarrow{R}^A_i) = f(\overrightarrow{R_i})$, for every $i = 1, 2, \dots, l'$.

In this situation we want to create $\mathcal{P}^A = (\mathbf{P}^A, \overrightarrow{R}^A)$ as a copy of $\mathcal{P}$ with $\mathcal{P}_i$ replaced by $\mathcal{P}^A_i$ for every $i = 1, \dots l'$. By (repeated) application of Lemma 3.2 we will then have $\mathcal{P}^A \sim \mathcal{P}$. To make this possible, we must show that no root vertex v of $\mathcal{P}_i$ is contained in some $\mathbf{P}_j \setminus R_j$ for $1 \leq i \leq l'$, $1 \leq j \leq l'$. (Otherwise replacing $\mathcal{P}_j$ by $\mathcal{P}^A_j$ may make it impossible to replace $\mathcal{P}_i$ by $\mathcal{P}^A_i$.) Assume, to the contrary that there is such a choice of $\mathcal{P}_i$, $\mathcal{P}_j$ and v. Because $\mathcal{P}_i$ is not subpiece of $\mathcal{P}_j$, nor vice versa, there is a root v' of $\mathcal{P}_i$ that is not contained in $\mathbf{P}_j$. Because v is in $\mathbf{P}_j \setminus R_j$ that is a g-component of $\mathbf{F}$ with g-cut R_j and v' is not, we conclude that v and v' are g-separated by R_j in $\mathbf{F}$. This leads to the fact that $R_j \cap P_i$ is g-cut in $\mathbf{P}_i$ g-separating v and v'.

This is not possible because, by our construction, $v, v' \in N_{\mathbf{P}}(P'_i)$ and $\mathbf{P}'_i$ is a g-component of $\mathbf{P}$ with g-cut $f^{-1}(C)$. Thus v and v' are connected by a walk in $G_{\mathbf{P}_i}$ containing only vertices of $\mathbf{P}'_i$. It is not possible for vertex of $\mathbf{P}'_i$ to be in R_j, because vertices of R_j are in $f^{-1}(C)$, while vertices of $\mathbf{P}'_i$ are in $f^{-1}(B \setminus C)$.

Finally define $f^A : P^A \to A$ as follows:

1. $f^A(x) = f_i^A(x)$ when $x \in P_i^A$ for some $i = 1, 2, \ldots, l'$;

2. $f^A(x) = f(x)$ otherwise.

It is easy to see that f^A is a homomorphism from $\mathbf{P}^A$ to $\mathbf{A}$ such that $f^A(\overrightarrow{R}^A) = (v_1, v_2, \ldots v_t)$. This is a contradiction with $(v_1, v_2, \ldots v_t) \notin X^k_{L(\mathbf{A})}$.

It remains to verify that $\mathbf{D} \in \mathrm{Forb_h}(\mathcal{F})$. We proceed analogously. Assume that f is a homomorphism from some $\mathbf{F} \in \mathcal{F}$ to $\mathbf{D}$ and further assume that the counter-example is chosen in a way so $\mathbf{F} \setminus f^{-1}(A)$ has minimal number of vertices.

Because $\mathbf{A}, \mathbf{B} \in \mathrm{Forb_h}(\mathcal{F})$, f must use vertices of $\mathbf{D} \setminus A$ and vertices of $\mathbf{D} \setminus B$ and, because $\mathbf{F}$ is connected, also vertices of $\mathbf{C}$. Analogously to the previous part, $f^{-1}(C)$ forms a g-cut in $\mathbf{F}$. Denote by $\mathbf{F}_A$ a g-component of $\mathbf{F}$ with g-cut $f^{-1}(C)$ contained in $\mathbf{F} \setminus f^{-1}(B)$ and by $\mathbf{F}_B$ a g-component of $\mathbf{F}$ with g-cut $f^{-1}(C)$ contained in $\mathbf{F} \setminus f^{-1}(A)$. Denote by R a minimal g-separating g-cut in $\mathbf{F}$ contained in $f^{-1}(C)$ that g-separate $\mathbf{F}_A$ and $\mathbf{F}_B$ (given by Proposition 2). Denote by $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ a piece of $\mathbf{F}$ containing $\mathbf{F}_B$. We have shown that $f(\overrightarrow{R}) \in X^i_{L(\mathbf{A})}$, for i such that $\mathcal{P} \in E_i$. Consequently there is $\mathcal{P}' = (\mathbf{P}', \overrightarrow{R}')$, $\mathcal{P} \sim \mathcal{P}'$, and a homomorphism $f^A : \mathbf{P}' \to \mathbf{A}$ such that $f^A(\overrightarrow{R}') = f(\overrightarrow{R})$. Now denote by $\mathbf{F}'$ a structure created from $\mathbf{F}$ by replacing piece $\mathcal{P}$ by piece $\mathcal{P}'$ and identifying $\overrightarrow{R}$ with $\overrightarrow{R}'$. Because $\mathcal{P} \sim \mathcal{P}'$, $\mathbf{F}'$ is isomorphic to some structure in $\mathcal{F}$. Now consider the homomorphism $f' : \mathbf{F}' \to \mathbf{D}$ defined as follows:

1. $f'(x) = f^A(x)$ for $x \in P'$,

2. $f'(x) = f(x)$ otherwise.

The size of $\mathbf{F}' \setminus f'^{-1}(A)$ is strictly smaller than the size of $\mathbf{F} \setminus f^{-1}(A)$, a contradiction with the minimality of the counter-example.

This finishes the proof of the amalgamation property of $\mathrm{Age}(\mathcal{L})$: while $\mathbf{V}$ may be infinite (because witness may be infinite), the lift $\mathbf{V}'$ induced on vertices $X \cup Y$ by $\mathbf{V}$ is the finite amalgamation of $\mathbf{X}, \mathbf{Y}$ over $\mathbf{Z}$ and thus $\mathbf{V}' \in \mathrm{Age}(\mathcal{L})$. $\square$

4 Non-existence of universal structures for classes $\mathrm{Forb_h}(\mathcal{F})$.

In this section we show that there exists infinite families $\mathcal{F}$ such that there is no universal structure for $\mathrm{Forb_h}(\mathcal{F})$. This is in contrast with the finite case, where the universal structures always exists.

Theorem 4.1 *Let $\mathcal{F}$ be a family of finite connected relational structures (of finite type). Assume that:*

(i) The size of all minimal g-separating g-cuts of structures in $\mathcal{F}$ is bounded by n.

(ii) Let $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ and $\mathcal{P}' = (\mathbf{P}', \overrightarrow{R}')$ be two pieces (of some structures in $\mathcal{F}$). Denote by $\mathcal{A}$ the rooted structure such that $\mathcal{P} \oplus \mathcal{A} = \mathbf{F} \in \mathcal{F}$. If $\mathcal{P}' \oplus \mathcal{A}$ is defined and $\mathcal{P}' \oplus \mathcal{A} \notin \mathrm{Forb_h}(\mathcal{F})$, then there is $\mathbf{F}' \in \mathcal{F}$ isomorphic to $\mathcal{P}' \oplus \mathcal{A}$.

Then the following conditions are equivalent:

(a) $\mathcal{F}$ is a regular family of connected structures.

(b) There is a ultrahomogeneous lift $\mathbf{U}'$ with only finitely many new relations such that its shadow $\mathrm{Sh}(\mathbf{U}')$ is a universal structure for the class $\mathrm{Forb_h}(\mathcal{F})$.

(c) There exists an ω-categorical universal structure for $\mathrm{Forb_h}(\mathcal{F})$.

Proof. $(a) \implies (b)$ follows from Theorem 3.1 for the class $\mathcal{F}$.

$(b) \implies (c)$ is immediate. The shadow of every ultrahomogeneous structure with finitely many relations is ω-categorical.

To see that $(c) \implies (a)$, assume to the contrary the existence of $\mathcal{F}$ satisfying (i) and (ii) which is not regular such that there is a universal structure $\mathbf{U} \in \mathrm{Forb_h}(\mathcal{F})$ which is ω-categorical.

Because the sizes of minimal g-separating g-cuts are bounded by n, we know that there is $n' \leq n$ with infinitely many pieces $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of arity n' such that the corresponding sets $\mathcal{I}_{\mathcal{P}_1}, \mathcal{I}_{\mathcal{P}_2}, \ldots$ are all different.

From Theorem 1.2 it follows that there are only finitely many orbits of n'-tuples. Denote by k the number of orbits of n' tuples. Now assign every piece $\mathcal{P}_i$ a set O_i of all orbits such that there exists a rooted homomorphism from $\mathcal{P}_i$ to $\mathbf{U}$ sending the root of $\mathcal{P}_i$ to the orbit. All the sets O_i are finite of size at most k. By the pigeonhole principle there is $i \neq j$ such that $O_i = O_j$.

By our assumption (ii) we know that for two pieces $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ and $\mathcal{P}' = (\mathbf{P}', \overrightarrow{R}')$ such that $\mathbf{P}, \mathbf{P}' \notin \mathrm{Forb_h}(\mathcal{F})$ we have $\mathcal{I}_{\mathcal{P}} = \mathcal{I}_{\mathcal{P}'}$ if and only if the rooted structure induced by $\mathbf{P}$ on $\overrightarrow{R}$ is identical to the rooted structure induced by $\mathbf{P}'$ on $\overrightarrow{R}'$. Consequently all those pieces belong to the same class of $\sim$ and there are only finitely many classes $\sim$ containing piece $\mathcal{P} = (\mathbf{P}, \overrightarrow{R})$ such that $\mathbf{P} \notin \mathrm{Forb_h}(\mathcal{F})$. We can thus assume that $\mathbf{P}_i, \mathbf{P}_j \in \mathrm{Forb_h}(\mathcal{F})$ and thus there is an isomorphic copy of both $\mathbf{P}_i$ and $\mathbf{P}_j$ in $\mathbf{U}$ (we where choosing i and j from infinitely many equivalence classes of $\sim$).

Now, because $\mathcal{I}_{\mathcal{P}_i} \neq \mathcal{I}_{\mathcal{P}_j}$ there is a rooted structure $\mathcal{A}$ that distinguishes $\mathcal{I}_{\mathcal{P}_i}$ from $\mathcal{I}_{\mathcal{P}_j}$. Without loss of generality assume that $\mathcal{A} \in \mathcal{I}_{\mathcal{P}_i}$. By our assumption (ii), $\mathcal{A} \oplus \mathcal{P}_j \notin \mathcal{F}$ implies $\mathcal{A} \oplus \mathcal{P}_j \in \mathrm{Forb_h}(\mathcal{F})$. Consequently there is an embedding from $\mathcal{A} \oplus \mathcal{P}_j$ to $\mathbf{U}$. This embedding must map the root of $\mathcal{A} \oplus \mathcal{P}_j$ to a tuple within an orbit $o \in O_j$. Since $\mathcal{A} \oplus \mathcal{P}_i \notin \mathrm{Forb_h}(\mathcal{F})$, we also have $o \notin O_i$. This is in contradiction to $O_i = O_j$. □

Example. The family $\mathcal{F}_{\mathrm{balanced}}$ of all balanced orientations of graph cycles (i.e., orientations having the same number of forward and backward edges) is not a regular family. Here all pieces are all oriented paths. The equivalence class of a piece depends on the algebraic length of the path (i.e., the number of the forward edges minus the number of backward edges) and there are infinitely many different algebraic lengths. Moreover $\mathcal{F}_{\mathrm{balanced}}$ satisfy the assumptions of Theorem 4.1 (the free amalgams of paths are cycles) and thus there is no ω-categorical universal structure for $\mathrm{Forb_h}(\mathcal{F}_{\mathrm{balanced}})$.

Further applications are given in the following section.

5 Homomorphism dualities and constraint satisfaction problems

A constraint satisfaction problem (CSP) is the following decision problem:

Instance: A finite structure $\mathbf{A}$.

Question: Does there exist a homomorphism $\mathbf{A} \to \mathbf{H}$?

We denote by $\mathrm{CSP}(\mathcal{H})$ the class of all finite structures $\mathbf{A}$ with $\mathbf{A} \to \mathbf{H}$ for some $\mathbf{H} \in \mathcal{H}$.

Recall that a *homomorphism duality* (for structures of given type) is any equation

$$\mathrm{Forb_h}(\mathcal{F}) = \mathrm{CSP}(\mathcal{H}).$$

When both $\mathcal{F}$ and $\mathcal{H}$ are finite sets of finite structures, we call the pair $(\mathcal{F}, \mathcal{H})$ *a finite duality pair* [16, 17, 11]. When $\mathcal{F}$ is an infinite set of finite structures, and $\mathcal{H}$ is a finite set of finite structures, we call it an *infinite-finite duality* [7].

Dualities play a role not only in complexity problems but also in logic, model theory, the theory of partial orders and category theory. In particular, it follows from [1] and [19] that dualities coincide with those first-order definable classes which are homomorphism-closed.

For the sake of simplicity, in the following discussion we shall restrict ourselves to the case where $\mathcal{D}$ consists of a single element $\mathbf{D}$. $\mathbf{D}$ is called the *dual of* $\mathcal{F}$ (it is easy to see that $\mathbf{D}$ is up to homomorphism-equivalence uniquely determined).

The notion of universal structures and duals is related. Given a class $\mathcal{K}$ of countable structures, an object $\mathbf{U} \in \mathcal{K}$ is called *hom-universal* for $\mathcal{K}$ if for every object $\mathbf{A} \in \mathcal{K}$ there exists a homomorphism $\mathbf{A} \to \mathbf{U}$. The following is immediate from the definitions:

Proposition 5.1 *Let $\mathcal{F}$ be a family of relational structures. Structure $\mathbf{D}$ is the dual of $\mathcal{F}$ if and only if $\mathbf{D}$ is hom-universal for* $\mathrm{Forb_h}(\mathcal{F})$.

In this section we shall show how to turn the universal structure constructed in Section 3 into a finite dual. This is possible only in the special cases where a finite dual exists. First we overview some results characterizing dualities.

A (relational) tree can be defined as follows (see [17]): The *incidence graph* $\mathrm{IG}(\mathbf{A})$ of relational structure $\mathbf{A}$ is the bipartite graph with parts A and $\mathrm{Block}(A)$, where

$$\mathrm{Block}(A) = \{(i, (a_1, \ldots, a_{\delta_i})) : i \in I, (a_1, \ldots, a_{\delta_i}) \in R^i_{\mathbf{A}}\},$$

and edges $[a, (i, (a_1, \ldots, a_{\delta_i}))]$ such that $a \in (a_1, \ldots, a_{\delta_i})$. (Here we write $x \in (x_1, \ldots, x_n)$ when there exists an index k such that $x = x_k$; $\mathrm{Block}(A)$ is a multigraph.) Relational structure $\mathbf{A}$ is called a *(relational) tree* when $\mathrm{IG}(\mathbf{A})$ is a graph tree (see e.g. [15]). The definition of relational trees by the incidence graph $\mathrm{IG}(\mathbf{A})$ allows us to use graph terminology for relational trees.

Theorem 5.1 ([17]) *For every finite family $\mathcal{F}$ of finite relational trees there exists a dual $\mathbf{D}$. Up to homomorphism-equivalence there are no other finite dualities with only one dual.*

Various constructions of duals of a given $\mathcal{F}$ are known [18]. More recently, infinite-finite dualities have been characterized:

Theorem 5.2 ([7]) *All regular families $\mathcal{F}$ of relational trees have a finite dual $\mathbf{D}$.*

Theorem 5.3 ([7]) *The family $\mathcal{F}$ of relational trees has a finite dual if and only if its upward closure* $\mathrm{UP}(\mathcal{F})$ *is regular.*

Here the *upward closure*, $\mathrm{UP}(\mathcal{F})$, is the class of all relational trees $\mathbf{T}_1$ such that there is $\mathbf{T}_2 \in \mathcal{F}$ and $\mathbf{T}_2 \to \mathbf{T}_1$.

We remark that all these characterizations extend naturally to duality pairs $(\mathcal{F}, \mathcal{D})$ where structures in the class $\mathcal{F}$ are not necessarily connected (i.e., they are relational forests). In this case however $\mathcal{D}$ generally consists of one or more structures. See [9, 7] for details.

The construction of Section 3 may be used to obtain an alternative way of constructing a dual in the proof of Theorems 5.1 and 5.2:

Corollary 5.1 (of Theorem 3.1) *Let $\mathcal{F}$ be a regular set of finite relational trees. Then there exists a class $\mathcal{L}$ of monadic lifts such that:*

(*i*) $\mathrm{Age}(\mathcal{L})$ *is an amalgamation class with free amalgamation;*

(*ii*) *The Fraïssé limit of* $\mathrm{Age}(\mathcal{L})$ *is an ultrahomogeneous structure* $\mathbf{U}'$ *such that* $\mathrm{Sh}(\mathbf{U}') = \mathbf{U}$ *is universal for* $\mathrm{Forb_h}(\mathcal{F})$*;*

(*iii*) $\mathbf{U}'$ *has a finite retract* $\mathbf{D}'$ *and consequently* $\mathrm{Sh}(\mathbf{D}') = \mathbf{D}$ *is a dual of* $\mathcal{F}$.

Proof. Observe that the minimal g-separating g-cuts of a relational tree all have size 1. Thus for a fixed family $\mathcal{F}$ of finite relational trees Theorem 3.1 establishes the existence of a class $\mathcal{L}$ and lift $\mathbf{U}'$ satisfying (*ii*). Class $\mathrm{Age}(\mathcal{L})$ is closed under strong amalgams that are free in the shadow. With only unary relations added to the lift, we immediately get that $\mathrm{Age}(\mathcal{L})$ is closed under free amalgamation, too, thereby obtaining (*ii*).

We show (*iii*). We find finite $\mathbf{D}'$ which is a retract of $\mathbf{U}'$, $\mathrm{Sh}(\mathbf{D}') \in \mathrm{Forb_h}(\mathcal{F})$, and for which there is a homomorphism $\mathbf{A} \to \mathrm{Sh}(\mathbf{D}')$ if and only if there is a homomorphism $\mathbf{A} \to \mathrm{Sh}(\mathbf{U}')$ for every relational structure $\mathbf{A}$.

Construct $\mathbf{D}'$ from $\mathbf{U}'$ by identifying all vertices of the same color (recall that the color of vertex v is the set $\{i; (v) \in X^i_{\mathbf{U}}\}$). Denote by r the homomorphism (retraction) $\mathbf{U}' \to \mathbf{D}'$. Obviously, if $f : \mathbf{Y} \to \mathbf{U}'$ is a homomorphism then $f \circ r : \mathbf{Y} \to \mathbf{D}'$ is also a homomorphism. Thus $\mathbf{A} \to \mathrm{Sh}(\mathbf{U}')$ implies $\mathbf{A} \to \mathrm{Sh}(\mathbf{D}')$.

It remains to show that $\mathrm{Sh}(\mathbf{D}') \in \mathrm{Forb_h}(\mathcal{F})$. Suppose, to the contrary, that there is a tree $\mathbf{F} \in \mathcal{F}$ and a homomorphism $f : \mathbf{F} \to \mathrm{Sh}(\mathbf{D}')$. Let $\mathbf{X}$ be a lift created from $\mathbf{F}$ by adding an extended relation $(v) \in X^i_{\mathbf{X}}$ if and only if $(f(v)) \in X^i_{\mathbf{D}'}$, for every $i = 1, 2, \ldots N$. Obviously f is also a homomorphism $\mathbf{X} \to \mathbf{D}'$. Consider lift $\mathbf{Y}$ induced by $\mathbf{X}$ on elements of some tuple $\vec{v} \in R^j_{\mathbf{X}}$. Since $\mathbf{F}$ is a relational tree, the shadow $\mathrm{Sh}(\mathbf{Y})$ has only one tuple. Because $\mathbf{D}'$

is retract of $\mathbf{U}'$ we know that the homomorphic image of $\mathbf{Y}$ is a substructure of $\mathbf{U}'$ and thus it is in $\mathcal{L}$.

Because $\mathbf{F}$ is a relational tree, it is possible to construct a homomorphic copy of $\mathbf{X}$ by starting with the homomorphic image of $\mathbf{Y}$ in $\mathcal{L}$ and using free amalgamation (over a one-element set) to add lifts of homomorphic images of all other tuples of $\mathbf{F}$. It follows that the homomorphic image of $\mathbf{X}$ is in $\mathcal{L}$, a contradiction. □

Remark. We stress the fact that families of trees are not the only regular families $\mathcal{F}$ of relational structures where the universal structure for $\mathrm{Forb_h}(\mathcal{F})$ can be described as a shadow of an ultrahomogeneous monadic lift $\mathbf{U}'$. For example, consider relational structures created from a relational tree by replacing tuples by an arbitrary irreducible structure (recall that a structure is *irreducible* if it has no vertex cuts). Such structures have all minimal g-separating g-cuts of size 1. One can easily construct continuum many such examples. There is however no finite retract of $\mathbf{U}'$ satisfying the statement of Corollary 5.1. (Such structures cannot be constructed from individual tuples by the aid of free amalgamation.) Of course such structures may have infinite chromatic numbers.

Note that it is also possible to construct the dual $\mathbf{D}$ of $\mathcal{F}$ without using the Fraïssé limit. Also this follows by our construction in Theorem 3.1. For every possible combination of new relations on a single vertex, create a single vertex of $\mathbf{D}$ and then keep adding tuples as long as possible so that $\mathbf{D}$ is still in $\mathcal{L}$ (in a similar way to the proof of Proposition 5.2).

We have shown that special cases of universal structures can be used to construct duals. Now we show the opposite: every dual can be turned into a universal structure by an especially simple monadic lift.

Proposition 5.2 *For a family $\mathcal{F}$ of relational structures the following two statements are equivalent:*

(i) There is a finite dual $\mathbf{D}$ of $\mathcal{F}$.

(ii) There exists a finite family $\mathcal{F}'$ of monadic lifts $\mathbf{X}$ whose shadow $\mathrm{Sh}(\mathbf{X})$ has one tuple with the following property:

Denote by $\mathcal{L}$ the class of all lifts $\mathbf{Y}$ such that

(a) $\mathbf{Y} \in \mathrm{Forb_h}(F')$, and

(b) every vertex of $\mathbf{Y}$ is in precisely one extended relation $X^i_{\mathbf{Y}}$.

There is a generic lift $\mathbf{U}'$ for $\mathcal{L}$ and its shadow $\mathrm{Sh}(\mathbf{U}')$ is an ω-categorical universal structure for $\mathrm{Forb_h}(\mathcal{F})$.

Loosely speaking, the class $\mathcal{L}$ is described by forbidden colors of vertices and forbidden colorings of edges.

Proof. $(i) \implies (ii)$: Fix a dual $\mathbf{D}$ with vertices $\{1, 2, \ldots N\}$ and consider lifts with N extended unary relations. Let $\mathcal{F}'$ be the family of all structures $\mathbf{X}$ such that:

1. the vertex set of $\mathbf{X}$ is $X \subseteq \{1, 2, \ldots N\}$;
2. there is a tuple $\vec{v} \in R^j_{\mathbf{X}}$ for some $j \in I$ such that $\vec{v} \notin R^j_{\mathbf{D}}$;
3. for every $i \in X$ there is a tuple $(i) \in X^i_{\mathbf{X}}$;
4. there are no other tuples in $\mathbf{X}$ and there are no vertices in X except ones in $\vec{v}$.

By definition, $\mathrm{Age}(\mathcal{L})$ is obviously an (free) amalgamation class (all forbidden substructures are irreducible).

We show that the shadow of $\mathcal{L}$ is $\mathrm{Forb_h}(\mathcal{F})$. For every $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$ and homomorphism $f : \mathbf{A} \to \mathbf{D}$ construct a lift $\mathbf{X}$ by putting $(v) \in X^i_{\mathbf{X}}$ if and only if $f(v) = i$. It is easy to see that $\mathbf{X} \in \mathcal{L}$. On the other hand, for every structure $\mathbf{A}$ and lift $\mathbf{X} \in \mathcal{L}$, a homomorphism $\mathbf{A} \to \mathrm{Sh}(\mathbf{X})$ can be interpreted as an $\mathbf{D}$-coloring of $\mathbf{A}$ and thus $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$.

The rest of statement follows by Fraïssé theorem analogously as Theorem 3.1.

In the opposite direction assume the existence of $\mathcal{F}'$, $\mathcal{L}$ and $\mathbf{U}'$ satisfying the statement of the proposition. Construct the retract $\mathbf{D}'$ of $\mathbf{U}'$ by unifying all vertices of the same color. This gives a homomorphism (retraction) $r : \mathrm{Sh}(\mathbf{U}') \to \mathbf{D}'$. Put $\mathbf{D} = \mathrm{Sh}(\mathbf{D}')$. We show that $\mathbf{D}$ is the dual of $\mathcal{F}$.

For every $\mathbf{A} \in \mathrm{Forb_h}(\mathcal{F})$ there is an embedding $e : \mathbf{A} \to \mathrm{Sh}(\mathbf{U}')$. It follows that $\mathbf{A} \in \mathrm{CSP}(\{\mathbf{D}\})$ because $e \circ r$ is a homomorphism $\mathbf{A} \to \mathbf{D}$. To see that $\mathbf{D} \notin \mathrm{Forb_h}(\mathcal{F})$, assume for a contradiction that there is $\mathbf{F} \in \mathcal{F}$ and a homomorphism $f : \mathbf{F} \to \mathbf{D}$. Create lift $\mathbf{X}$ from $\mathbf{F}$ by adding a tuple (v) to $X^i_{\mathbf{X}}$ if and only if $(f(v)) \in X^i_{\mathbf{D}'}$. Lift $\mathbf{X}$ satisfy condition (a) of the definition of $\mathcal{L}$. For every $\mathbf{F}' \in \mathcal{F}$ a homomorphism $\mathbf{F}' \to \mathbf{X}$ implies a homomorphism $\mathbf{F}' \to \mathbf{U}$ giving (b) and thus $\mathbf{X} \in \mathcal{L}$. A contradiction with $\mathbf{U}'$ being generic for $\mathcal{L}$ and $\mathrm{Sh}(\mathbf{U}') \in \mathrm{Forb_h}(\mathcal{F})$. $\square$

Theorem 4.1 and Proposition 5.2 imply Theorem 5.3. It is easy to see that $\mathrm{UP}(\mathcal{F})$ have the size of g-separating g-cuts o bounded by 1 and moreover it is closed for free amalgams of trees.

To show both implications of Theorem 5.1 it is necessary to show the non-existence of monadic lifts as described in Proposition 5.2 for families $\mathcal{F}$

not consisting of relational trees. This is possible with a more systematic study of the minimal arities needed in the lift for a given family $\mathcal{F}$, and by giving a more explicit description of the lifts via forbidden substructures, as shown in [13].

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References

[1] A. Atserias: On digraph coloring problems and tree with duality. In: LICS'05, IEEE (2005), 106–115.

[2] B. Bolobás: Modern Graph Theory, Graduate Texts in Mathematics, vol. 184, Springer, 1998.

[3] G. L. Cherlin, S. Shelah, N. Shi: Universal Graphs with Forbidden Subgraphs and Algebraic Closure, Advances in Applied Mathematics 22 (1999), 454–491.

[4] G. L. Cherlin, N. Shi: Graphs omitting a finite set of cycles, J. Graph Theory 21 (1996), 351–355.

[5] J. Covington: Homogenizable Relational Structures, Illinois J. Mathematics, 34(4) (1990), 731–743.

[6] E. Engeler: Äquivalenz von n-Tupeln, Z. Math. Logik Grundl. Math. 5 (1959), 121–131.

[7] P. L. Erdős, D. Pálvölgyi, C. Tardif, G. Tardos: On infinite-finite tree-duality pairs of relational structures, arXiv:1207.4402v1 (submitted) (2012).

[8] P. L. Erdős, C. Tardif, G. Tardos: On infinite-finite duality pairs of directed graphs, arXiv:arXiv:1207.4402v1 (subitted) (2012).

[9] J. Foniok, J. Nesetril, C. Tardif: Generalised dualities and maximal finite antichains in the homomorphism order of relational structures. Eur. J. Comb. 29(4): 881-899 (2008)

[10] R. Fraïssé: Théorie des relations, North Holland, 1986.

[11] P. Hell, J. Nešetřil: Graphs and homomorphisms, Oxford University Press (2004).

[12] W. Hodges: Model Theory, Cambridge University Press (1993).

[13] J. Hubička, J. Nešetřil: Homomorphism and embedding universal structures for restricted classes of graphs, arXiv:0909.4939v1 (2009).

[14] G. Kun, J. Nešetřil: NP by means of lifts and shadows, MFCS'07 LNCS 4708 (2007), 171–181.

[15] J. Matoušek, J. Nešetřil: Invitation to discrete mathematics, Oxford Univ. Press (1998), 2nd edition (2007).

[16] J. Nešetřil, A. Pultr: On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22 (1978), 287–300.

[17] J. Nešetřil, C. Tardif: Duality Theorems for Finite Structures (Characterizing Gaps and Good Characterizations), J. Combin. Theory B, 80 (2000), 80–97.

[18] J. Nešetřil, C. Tardif: Short answers to exponentially long questions: Extremal aspects of homomorphism duality, SIAM J. Discrete Math. 19, No. 4 (2005), 914–920.

[19] B. Rossman: Existential positive types and preservation under homomorphisms. J. ACM 55 (3) (2008), 1–53.

[20] C. Ryll-Nardzewski, One the categoricity in power $\aleph_0$, Bull. Acad. Polon. Sci. Ser. Math. 7 (1959), 545–548.

[21] N. Svenonius: $\aleph_0$-categoricity in first-order predicate calculus, Theoria 25 (1955), 82–94.