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Josef Cibulka (ed.)

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Preface

Charles University in Prague and particularly Department of Applied Mathematics (KAM), Computer Science Institute of Charles University (IUUK) and its international centre DIMATIA, are very proud that they are hosting one of the very few International REU programmes which are funded jointly by NSF and the Ministry of Education of Czech Republic (under the framework of Kontakt programmes ME 521, ME 886 and ME 09074). This programme is a star programme at both ends and it exists for more than a decade since 2001. Repeatedly, it has been awarded for its accomplishments and educational excellence.

This booklet reports just the programme in 2012. I thank to Josef Cibulka, the Czech mentor of this year, for a very good work both during the programme itself and after.

Prague, November 11, 2012

Jaroslav Nešetřil

DIMACS/DIMATIA Research Experiences for Undergraduates (REU) is a joint program of the DIMATIA center, Charles University in Prague and DIMACS center, Rutgers University, New Jersey. This year's participants from Charles University were students Martin Balko, Ondřej Bílka, Martin Böhm and Pavel Veselý. Their coordinator was Josef Cibulka, who participated in the scientific work, but mainly took care of organizing the DIMATIA part of the program. Together with more than thirty students from universities from all over the United States, they participated in the first part of the program, at Rutgers University of New Jersey in Piscataway, USA, from June 3rd to July 20th. Five American students were selected to join, together with their coordinator, the Czech students in the second part which took place at Charles University in Prague from July 24th to August 8th. The students were Marissa Loving, Theresa Lye, Michael Poplavski, Ixtli-Nitzin Sanchez and Ethan Schwartz. The coordinator was Kellen Myers.

The first part of the program mainly consists of students solving open mathematical problems brought by their mentors. Students attended several lectures including a lecture by Professor Nešetřil called "What Makes a Math Problem Beautiful?". DIMACS organized trips to the IBM Watson Research center and the Cancer Institute of New Jersey. Here the students heard about applications of mathematics and computer science and were given excursions around the facilities.

In Prague, the students attended a series of lectures given by professors mainly from the Department of Applied Mathematics and the Computer Science Institute of Charles University. They also had the opportunity to attend the Midsummer Combinatorial Workshop.

In addition to the scientific program, an important part of the REU is an intercultural experience. During the first part, an afternoon was dedicated to presentations of Czech Republic and cultures from which the American students come from. The students participated together in informal sport activities and sightseeing trips.

The students got important experiences with research and life abroad. For some of them, the program will certainly be an important milestone in their future scientific career.

This booklet presents the results of the Czech students stemming from the REU programme and reports of the American students about their visit to Prague.

Josef Cibulka

The participants of the Prague part of the programme.

Midsummer Combinatorial Workshop excursion to the baroque library of the Strahov Monastery.

Contents

Almost Monochromatic Partitions of Two-Colored Planar Points Sets

Martin Balko and Martin Böhm

Suppose that $S = (W, B) \subset \mathbb{R}^2$ is a set of points in general position (thus no three points from S are collinear) colored with two colors – white and black. Let $\sigma(S, k)$ denote the minimum number of subsets in a partition of $S = (W, B)$ such that their convex hulls are pairwise disjoint and all the points in each subset have the same color except at most k of them (the *stains*). Let

$$\sigma(n, k) = \max\left\{\sigma(S, k) \mid S \subset \mathbb{R}^2, |S| = n\right\}.$$

For $k = 0$, the $\sigma(S, 0)$ denotes the cardinality of the smallest partition of S into monochromatic subsets with pairwise disjoint convex hulls. Dumitrescu and Pach [2] already showed that

$$\sigma(n, 0) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Grima et al. [4] also consider the function $\sigma(n, k)$ and they show the following bounds:

$$\left\lceil \frac{n+1}{4} \right\rceil \leq \sigma(n, 1) \leq \begin{cases} 3\left\lfloor \frac{n}{11} \right\rfloor & \text{if } n' = 0 \\ 3\left\lfloor \frac{n}{11} \right\rfloor + \left\lceil \frac{n'+1}{4} \right\rceil & \text{if } n' \neq 0 \end{cases}$$

where n' is a residue of dividing n by 11. We would like to derive some more bounds for the function $\sigma(n, k)$ for general $k \in \mathbb{N}_0$. More results related to the various forms of balanced subdivisions of two-colored planar point sets can be found, for example, in the survey by Kaneko and Kano [7].

The upper bound

$$\sigma(n, k) \leq \left\lceil \frac{n}{2k+1} \right\rceil$$

holds trivially, as every set of $2k + 1$ bichromatic points contains at most k points of a single color. Modifying the proof of Grima et al. [4] we can show that this bound is actually quite accurate.

Theorem 1.1. *For every $k \in \mathbb{N}_0$, $\sigma(n, k) \geq \left\lceil \frac{n+1}{2k+2} \right\rceil$.*

Proof. Let n be even and let $S = (W, B)$ be a convex set of n points such that for every vertex v the colors of the neighbors of v on the boundary of the convex hull differ from the color of v. Suppose that Π is a partition of S with minimum number of sets. We define a rooted tree T with vertex set Π. A subset B_i is a descendant of B_s if $conv\,(B_i \cup B_s)$ does not intersects any other sets of Π. A subset B_j is a descendant of $B_i \neq B_s$ if B_i is a descendant of B_h, B_j is not descendant of B_h and $conv\,(B_i \cup B_j)$ does not intersects any other sets of Π. As a root of T, we choose a set B_s which contains at least $2k+2$ points (such set exists, as otherwise the bound holds trivially).

Let n_i denote the number of interior nodes of T which contain i points and let h denote the number of leaves of T. Every interior node of T corresponds to a sets of Π which contains at least two points (otherwise it does not have a descendant). Every leaf of T corresponds to a set which has at most $2k+1$ points, as the set can contain part of the boundary of the convex hull of S which contains $2k+1$ vertices. We assign every leaf to a unique interior node of T recursively. First, we assign at least $|B_s| - 2k$ leaves to B_s, as no set contains more than $2k$ edges of the convex hull of S (consider an alternating path).

If $l \geq 2k+3$, then for every B_p, $B_p \neq B_s$ and $|B_p| = l$, we assign at least $l - 2k - 2$ leaves. Since there are at least $l - 2k - 1$ outgoing edges from B_p (consider the edges inside the convex hull of S) and one of them is assigned to the predecessor of B_p. We thus have

$$h \geq n_{2k+3} + 2n_{2k+4} + \cdots + (t - 2k - 2)\, n_t + 2$$

where t is the number of points in the largest set of Π (two additional leaves are added, as the root B_s has no predecessor).

On the other hand we have

$$h \geq \frac{n - 2n_2 - 3n_3 - \cdots - tn_t}{2k+1},$$

since each leaf contains at most $2k+1$ points and points in the interior nodes are not in the leaves. Combining these inequalities we get

$$(2k+2)\, h \geq n - 2n_2 - 3n_3 - \cdots - (2k+2)\, n_{2k+2} - \cdots - (2k+2)\, n_t + 2.$$

Therefore we have

$$(2k+2)\, h \geq n - (2k+2)\,(n_2 + \cdots + n_t) + 2.$$

This implies

$$\sigma(S,k) \geq h + n_2 + n_3 + \cdots + n_t + 2 \geq \left\lceil \frac{n+2}{2k+2} \right\rceil.$$

The case when n is odd can be treated the same way (just remove one point from S). $\square$

To derive stronger upper bounds we use the famous Ham Sandwich theorem which says that every two-colored set of points in general position in the plane can be partitioned by a line such that every open halfplane spanned by this line contains approximately half of the points of each color.

Theorem 1.2 (the Ham Sandwich theorem, [8])**.** *Let $A_1, A_2, \ldots, A_d$ be sets of points in $\mathbb{R}^d$ such that $A_1 \cup A_2 \cup \ldots \cup A_d$ is in general position. Then there exists a hyperplane h such that each open halfspace contains exactly $\lfloor \frac{1}{2} |A_i| \rfloor$ points of A_i.*

Lemma 1.3. *For every $k \in \mathbb{N}_0$, $\sigma(4k+3, k) \leq 2$.*

Proof. Let $S = (W, B)$ be two colored point set with $4k+3$ points and suppose that $|W| > |B|$. Hence $|B| \leq 2k+1$. If $|B| \leq 2k$, then we use the Ham Sandwich theorem which partitions W and B into two open halfplanes such that each contains at most k black vertices.

So we may assume that $|B| = 2k+1$. If we remove an arbitrary point $w \in W$, then we get the set S' with $2k+1$ black and $2k+1$ white vertices. The Ham Sandwich theorem partitions those sets by a line l such that there are k black and k white vertices in each open halfplane (h^+ and h^-) and one black vertex y and white vertex z lying on l. Since S is in general position, the point w does not lie on l. Without loss of assumption suppose that $w \in h^+$. Then we have the desired partition $(S \cap h^+) \cup \{z\}$, $(S \cap h^-) \cup \{y\}$ of S. See the following figure (we use the notation (w, b) for a set with w white and b black points). $\square$

Corollary 1.4. *Every two colored set of n points can be partitioned into at most $2\lceil \frac{n}{4k+3} \rceil$ sets such that their convex hulls are pairwise disjoint and each subset has at most k stains.*

Proof. Consider partitioning of the set S by lines into parts with $4k+3$ points (except for the last part which can contain smaller number of points). Then use the previous lemma for each part. $\square$

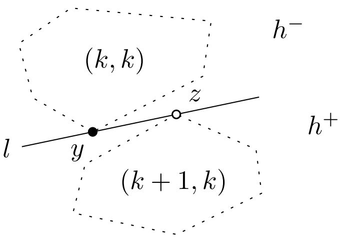

Figure 1: Using the Ham Sandwich theorem

Corollary 1.5. *For every* $k \in \mathbb{N}_0$, $\left\lceil \frac{n+1}{2k+2} \right\rceil \leq \sigma(n,k) \leq 2\lceil \frac{n}{4k+3} \rceil$.

To prove some other results we use auxiliary results concerning the dissection graphs of planar point sets proved by Erdős et al. [3] Let S be set of n points in general position in plane. For every two points p and q from S, let $N(\overrightarrow{pq})$ denote the number of points from S in the open halfplane on the right side of the directed line $\overrightarrow{pq}$. A *dissection l-graph G_l of planar point set S* is a directed graph whose edges are the segments $\overrightarrow{pq}$ with $N(\overrightarrow{pq}) = l$. Since $G_{n-l-2} = -G_l$ (we use $-G$ to denote the graph G with reversed orientation of edges), it suffices to consider the cases $0 \leq l \leq \frac{n-2}{2}$.

Theorem 1.6 ([3])**.** *Each component of G_l has an oriented Eulerian cycle.*

Theorem 1.7 ([3])**.** *Every point of S is a vertex of $G_{\lfloor \frac{n-2}{2} \rfloor}$. If $l < \lfloor \frac{n-2}{2} \rfloor$, then the graph G_l has at least $2l+3$ vertices.*

Proposition 1.8. *For every $k \in \mathbb{N}_0$ and every set $S = (W,B)$ of $6k+5$ points in general position colored with two colors such that $|W| - |B| = 3$ the inequality $\sigma(S,k) \leq 3$ holds.*

Proof. Let S be such planar point set. Then $|W| = 3k+4$ and $|B| = 3k+1$. Consider the dissection graph G_l for S where $l = 2k+1$. Since the number of vertices of G_l is at least

$$2l + 3 = 2(2k+1) + 3 = 4k + 5,$$

we know that there are both white and black vertices in G_l. As each component of G_l contains an oriented Eulerian cycle, there is an oriented edge in G_l with either both vertices white or with one vertex black and the other one white.

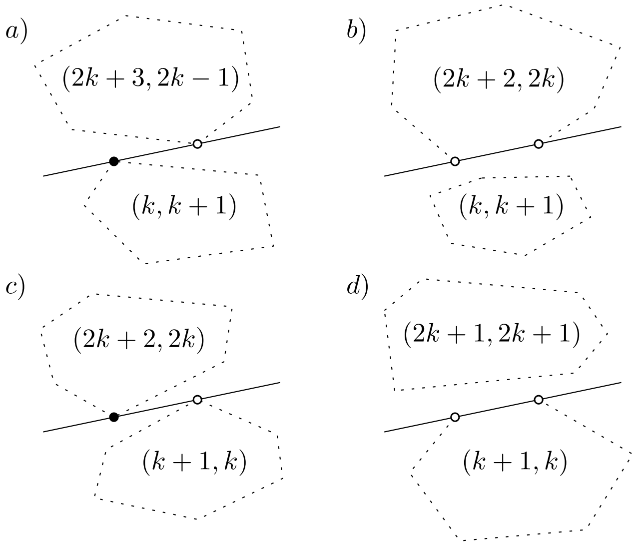

Figure 2: Creating the partition

Consider a line l which corresponds to such edge. Then l separates $2k+1$ points from S. If there are at most $k-1$ points of a single color between the separated points, then we can always add one point lying on l to the separated set. Thus there are at most $4k+3$ points in the rest of S and we know that it can be partitioned into two sets, according to our previous proposition. This gives us three sets in the final partition.

Thus it remains to solve the case when each color contains at least k separated points. See the figure how to handle this situation. In three of these cases (a), c) and d)) we use the first proposition again, since there are at most $4k+3$ unseparated points, and in the remaining one (case b)) we can use the halving line to partition the unseparated points of S into two sets. $\square$

Thus to prove the following conjecture, it suffices to derive the bound for the two colored sets S where the number of white and black points differ by one, as the case $|B| \leq 3k$ is not difficult (consider two lines, each separating a part of S with three black points). If such statement is true, then it implies the result of Grima et al. [4] for $k = 1$.

Conjecture 1.9. *For every* $k \in \mathbb{N}_0$, $\sigma(n,k) \le 3\left\lceil \frac{n}{6k+5} \right\rceil$*?*

For integers $w \ge 0$ and $b \ge 0$, we call a line which intersects two points of S a (w,b)*-cut*, if it separates w white and b black points of S. Similarly we say that such line is l*-cut*, $l \ge 0$, if it separates l points of S (no colors specified). The cut is white (black) if both points lying on such line are white (black, respectively). The cut is called bichromatic if the line contains both black and white point. We also say that two-colored set is of *type* (w,b), if it contains w white and b black points.

Suppose that we have a two-colored point set $S = (W,B)$ with $|W| = 3k+3$ and $|B| = 3k+2$ which cannot be partitioned into three monochromatic subset with at most $k \ge 0$ stains and pairwise disjoint convex hulls. We show some properties that S must satisfy. However we have not been able to prove the nonexistence of such sets in the general case yet.

Proposition 1.10. *The dissection graph* G_{2k+1} *of* S *contains only monochromatic components. If* S' *is a set of points which remain in* S *after deletion of points separated by an arbitrary white* $2k+1$*-cut, then the dissection graph* G'_{2k+1} *of* S' *does not contain any white edge.*

Proof. In fact we prove a slightly stronger statement. Consider an arbitrary $(2k+1)$-cut l. If the set of points separated by l contains less than k points of a single color, then S can be separated, as we can always add at least one point lying on l between the separated ones and use Lemma 1.3 on the rest. Thus l separates either the set of type $(k+1,k)$ or $(k,k+1)$.

There are six different cases (see the figure bellow) and only two of them do not cause separation of S – white $(k,k+1)$-cut (case a)) and black $(k+1,k)$-cut (case f)). Thus the components of the dissection graph G_{2k+1} are monochromatic. We can also examine the set of unseparated points in the same way.

After performing the white cut, we obtain set of type $(2k+3, 2k+1)$. The only cuts which do not separate such set into two parts are black $(k+1,k)$-cut and bichromatic $(k+1,k)$-cut. Thus the corresponding dissection graph does not contain a white edge. On the other hand, if we perform black $(k+1,k)$-cut on S, then the unseparated points form set of type $(2k+2, 2k+2)$. Then the only two interesting cases are, again, white $(k,k+1)$-cut and black $(k+1,k)$-cut, hence there is no bichromatic $(2k+1)$-cut. □

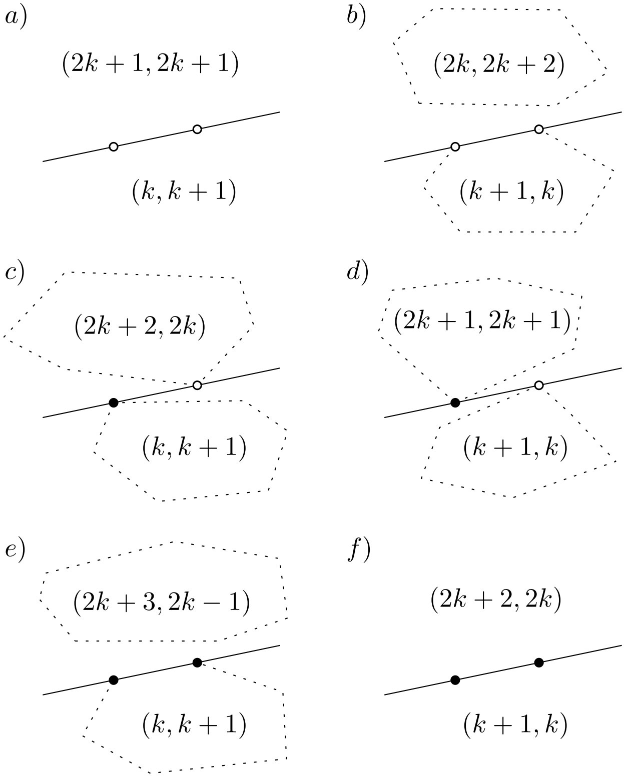

Figure 3: Possible $(2k + 1)$-cuts

Proposition 1.11. *The set S has only black points on the boundary of its convex hull.*

Proof. If we have a white point w on the boundary of the convex hull of S, then we consider two $(2k+1)$-cuts containing w. According to the previous proposition, these cuts are both white (otherwise S can be partitioned). But then the dissection graph of unseparated points contains a white edge, which is a contradiction. $\square$

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On the complexity of minimizing the number of edge contractions to make a graph planar

Ondřej Bílka

We study a problem to determine the minimal number of edge contractions to make a graph planar. We show that this problem is NP complete. We do a reduction to planar Steiner tree problem.

Let G be a connected planar graph and S a set of vertices that need to be connected by Steiner tree. We define graph G' such that minimal number of contractions to make G planar is the size of minimal Steiner tree of G.

G' consist of three parts. First part is copy of G. Second part consist of for vertices x_1, x_2, x_3, y_1. For third part first consider complete bipartite graph B between $\{x_1, x_2, x_3\}$ and $S \cup y_1$. Subdividing each edge of B into path of lenght $|G| + 1$ yield graph G'.

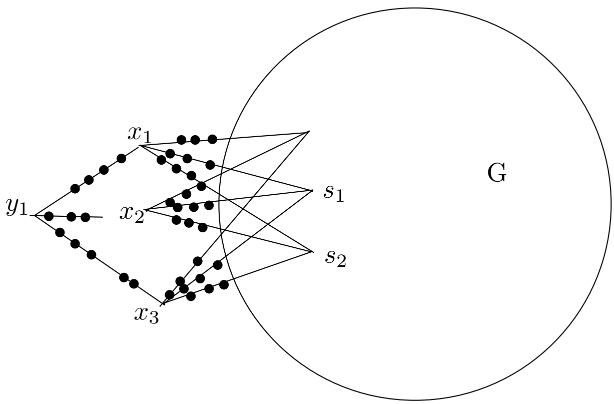


Figure 1

If T is a Steiner tree for a given pair G, S then contracting edges of T in G' gives us copy of $G \backslash T$ which is planar with subdivided $K_{2,3}$ attached to a contracted vertex. Thus $G' \backslash T$ is planar.

Now assume that we have minimal set of edges C whose contraction makes G' planar. Observe that C must be a forest. Distance of points x_1, x_2, x_3, y_1 from each other and from S is at least $|G| + 1$. As contracting entire G makes G' planar we know that minimal C must consist only of edges from G. If C contains connected component containing S we found a Steiner tree. Otherwise let C_1, C_2 be components of C that contain a point from S. Then graph $G' \setminus C$ contains vertices c_1, c_2 obtained by contracting C_1, C_2. Vertices x_1, x_2, x_3 and c_1, c_2, y_1 form subdivided $K_{3,3}$ which contradicts planarity of $G' \setminus C$.

Holes in bicolored random point sets

Josef Cibulka and Jan Kynčl

All point sets considered in this paper are in the plane and in general position, that is, no three of them lie on a line. We are given a set P of points in the plane. A *polygon on a set P' of points* is a simple polygon with its vertices embedded on the points of P'. A polygon $\mathcal{P}$ on $P' \subset P$ is a *general hole* in P if the interior of $\mathcal{P}$ does not contain any point of P. A *convex hole* is a convex polygon that is a general hole. The *size of a hole* is the number of its vertices.

Erdős and Szekeres showed in 1935 [4] that for every k every large enough set of points in the plane contains a k-tuple of points in convex position. Erdős proposed a question whether this remains true if we require the k-tuple to form a convex hole. This question was answered in the negative by Horton [5]. See for example the book by Harris, Hirst and Mossinghoff [6] for more results.

A *bicolored point set* is a set of points some of which are white and others black. A hole is *monochromatic* if all its vertices have the same color. Devillers et al. [3] studied the maximum size of a convex hole guaranteed to exist in every bicolored set of n points. They showed that a monochromatic triangle exists for sufficietly large n, and that for every n, there is a bicolored set of n points with no monochromatic convex 5-hole. The question of the existence of a monochromatic convex 4-hole remains open. Aichholzer et al. [1] showed that a general monochromatic 4-hole exists in every sufficiently large bicolored point set.

For brevity, writing that P is a random set of n points from a region $\mathcal{R} \subset \mathbb{R}^2$ means that P is created by the process of selecting n points independently and uniformly at random from $\mathcal{R}$. Notice, that if $\mathcal{R}$ is a bounded convex region of positive measure, then such a point set is in general position with probability 1. A balanced 2-colored random set P of n points is $P = W \cup B$, where B is a random set of $\lfloor n/2 \rfloor$ black points and W is a random set of $\lceil n/2 \rceil$ white points. Let $\mathcal{R}$ be an arbitrary bounded convex region of positive measure. Balogh et al. [2] proved that the expected maximum size of a convex hole in a random set of n (uncolored) points from $\mathcal{R}$ is $\Theta(\log n / \log \log n)$. This implies that the expected maximum size of a monochromatic convex hole in a 2-colored random set of n points from $\mathcal{R}$ is $\Theta(\log n / \log \log n)$.

We study general holes in random bicolored point sets. In Theorem 1.2, we show that a random set of bicolored points from a convex region contains a monochromatic hole of size $\Omega(\log(n))$ with high probability. In Theorem 1.3, we show that a random set of cm white and m/c black points from a unit square contains a white hole of size $\Omega(m)$ with high probability.

Observation 1.1. *Let $Q \subset \mathbb{R}^2$ be a set of at least three points in general position and let e be a fixed edge of $conv(Q)$. Then there is a polygon $\mathcal{P}$ with the vertex set Q and having e as an edge. Notice that $\mathcal{P}$ is contained in $conv(Q)$.*

Proof. Let $n = |Q|$. Label the two vertices of e q_1 and q_2 so that q_1 immediately precedes q_2 when going along the convex hull in the clockwise order. We label the points of Q $q_1, \ldots, q_n$ in such a way that $q_2, \ldots, q_n$ are listed in the clockwise order of visibility from q_1. The polygon $\mathcal{P}$ is the polygon with vertex set Q that visits the vertices in the order $q_1, \ldots, q_n$. □

Let X_n be a nonnegative random variable depending on a parameter n and let $f : \mathbb{R} \to \mathbb{R}$ be a function. If there are constants $c_1, c_2 > 0$ such that for every $n \in \mathbb{R}$, $|X_n - f(n)| \leq |f(n)|/2$ with probability at least $1 - c_1 n^{-c_2}$, then we say that X_n is in $\Theta(f(n))$ *with high probability*. Similarly, $X_n \in \Omega(f(n))$ with high probability if $X_n \geq f(n)/2$ with probability at least $1 - c_1 n^{-c_2}$.

Theorem 1.2. *Let $\mathcal{R} \subset \mathbb{R}^2$ be a bounded convex region of positive measure. With high probability, the maximum size of a general monochromatic hole in a balanced 2-colored random set of n points from $\mathcal{R}$ is $\Omega(\log(n))$.*

Proof. Let R_n be a balanced 2-colored random set of n points from $\mathcal{R}$. We assume that the x-coordinates of the points in R_n are pairwise different, which occurs with probability 1. We now count the probability that the sequence of colors of the points of R_n sorted by the x-coordinate has no white subsequence of length $\log(n)/2$. We consider $\lfloor n/(2^{-1}\log n)\rfloor$ disjoint sequences of $\log(n)/2$ consecutive points each. Each of these sequences contains only white points with probability $2^{-\log n/2}$. Thus the probability that there is no white subsequence of length $\log(n)/2$ is at most

$$(1 - 2^{-\log n/2})^{\lfloor n/(\log(n)2^{-1})\rfloor} \leq e^{-n^{-1/2}n/\log(n)} = n^{-n^{1/2}\log(e)/\log^2(n)}$$

for n large enough. Therefore, with high probability, there are $\log(n)/2$ white points whose convex hull contains no black point. By Observation 1.1, we can find an empty white polygon with all these white points as vertices. □

In the following, we do not aim on optimizing the ratio between the number of white and black points. However, it seems unlikely that the proof could be easily extended to balanced 2-colored point sets.

Theorem 1.3. *There are constants c and m_0 such that the following holds. Let $m \geq m_0$, where m is of the form $m = k^2$ for some integer k. Let $P = W \cup B$, where W is a random set of cm white points and B a random set of m/c black points from the unit square. Then P contains a white hole of size at least $\Omega(m)$ with high probability.*

Proof. We cut the unit square by horizontal and vertical lines into a grid of $k \times k$ square cells of side length $1/k$. The (i, j)-cell is the cell in the i-th row from top and j-th column from the left. Each cell is subdivided into 4 quadrants of side length $1/(2k)$. With probability 1, every point of $B \cup W$ lies in the interior of some quadrant of some cell. A cell is *useful* if it contains a white point in every quadrant and no black point.

We view the grid as a graph, where each cell is connected to the at most four cells with which it shares an edge.

Let p_c be the node percolation constant for the square grid. Experimental results suggest that p_c is around 0.6. A classical result in the percolation theory states that for every $p > p_c$ there exists a constant k_0 such that whenever $k \geq k_0$, the induced subgraph of the $k \times k$ square grid obtained by taking each node with probability p has a connected component of size $\Omega(k^2)$ with high probability. See for example the book of Penrose [7].

Let c be a constant satisfying $4e^{-c/4} + 1/c < 1 - p_c$. Assuming $p_c \leq 0.6$, we can use $c = 10.4$. The probability that a given cell is not useful is at most

$$4\left(1 - \frac{1}{4m}\right)^{cm} + \frac{1}{c} \leq 4e^{-c/4} + \frac{1}{c} < 1 - p_c.$$

The first summand evaluates the probability that the cell has no white point in one of its quadrants. The second summand is an upper bound on the probability that the cell contains a black point.

We take such a constant m_0 that the subgraph of the grid graph induced by the useful cells contains a connected component with $\Omega(m)$ vertices with high probability.

Assuming the induced graph contains the component with $\Omega(m)$ vertices, we let G be the graph induced by the vertices of this component. Let V be the set of nodes of G. We fix a spanning tree T of G. Let Q_T be the set of $4|V|$ white points with one white point from each quadrant of each cell corresponding to a vertex in V.

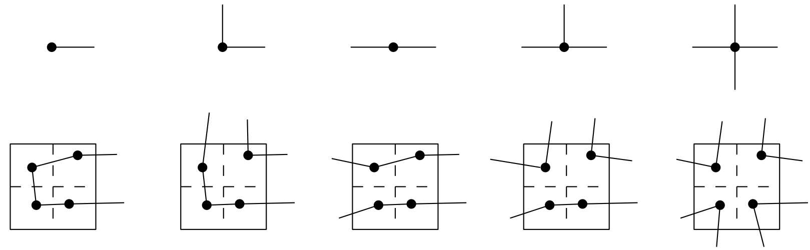


Figure 1: Top: All the possible sets of neighbors of a node in T (up to symmetries). Below: Connections between the white points in the quadrants of the node.

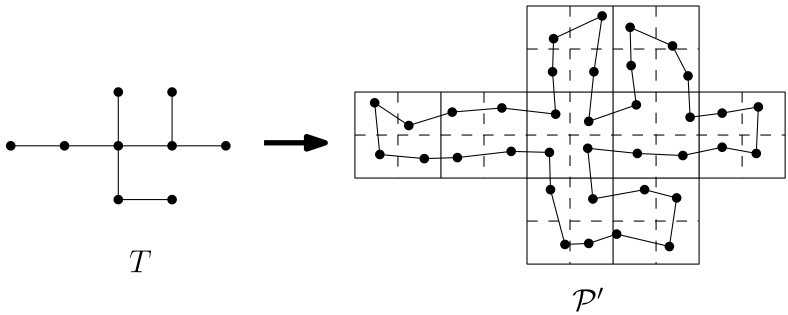
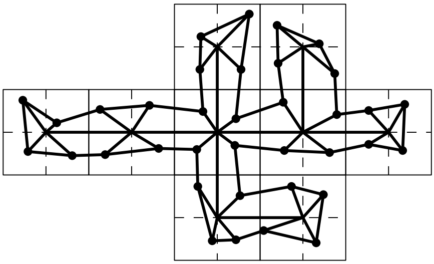


Figure 2: Example of a polygon $\mathcal{P}'$ created from a tree T.

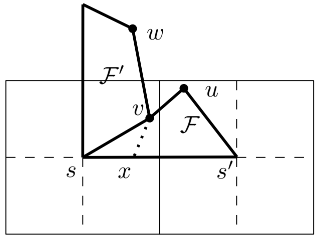
We create a polygon $\mathcal{P}'$ by connecting the white points in each node depending on the set of its neighbors in the tree T. See Figs. 1 and 2.

The polygon $\mathcal{P}'$ is contained within the union of useful cells and so it does not contain any black point. We need to consider the possible white points in its interior. For this purpose, for each vertex v of $\mathcal{P}'$, we draw a segment between v and the center of the cell in which v is contained. We also draw segments between the pairs of centers of cells that correspond to nodes connected by an edge in T. These segments split $\mathcal{P}'$ into a collection $\mathcal{C}$ of triangles and quadrangles. See Fig. 3. In addition, one edge of each of these triangles and quadrangles is an edge of $\mathcal{P}'$.

If some polygon in $\mathcal{C}$ is not convex, then it is a quadrangle with an angle of degree larger than 180° at a vertex of $\mathcal{P}'$. Let $\mathcal{F}$ be the nonconvex quadrangle with an angle larger than 180° at vertex v. Let $\mathcal{F}'$ be the other



a)



b)

Figure 3: a) Splitting $\mathcal{P}'$ into convex regions. b) Fixing a non-convex polygon; dotted segment replaces the segment sv.

polygon in $\mathcal{C}$ that has v as a vertex. Let u and w be the white points that are vertices of $\mathcal{F}$ and $\mathcal{F}'$ different from v. Let s and s' be the centers of the cells containing v and u, respectively. We replace the segment sv by the segment xv, where x is a point on the segment ss' such that the angles in which v sees the pairs w, s and s, u are smaller than 180°. See Fig. 3.

The polygon $\mathcal{P}'$ is now split into a set $\mathcal{C}'$ of convex polygons. Each polygon $\mathcal{F}$ in $\mathcal{C}'$ has an edge e of $\mathcal{P}'$ as its edge. For every polygon $\mathcal{F}$ containing at least one white point in its interior, we make the following. We apply Observation 1.1 on the set of white points in $\mathcal{F}$, including the two white points on the boundary of $\mathcal{F}$. We consider the curve obtained by removing the edge e from the polygon guaranteed by the observation. We replace the edge e in $\mathcal{P}'$ by this curve.

The resulting polygon $\mathcal{P}$ has no point of $B \cup W$ in its interior and has at least $4|V|$ vertices.

□

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Peel numbers

Pavel Veselý

Let $G = (V, E)$ be a simple graph. We define *peel number* of a vertex by a computation process in which we delete (peel) vertices one by one. The computation consists of phases. In the phase zero we delete vertices with degree zero and assign them peel number zero. In the phase one we delete vertices with degree one or less recursively, i.e. a vertex that has originally degree more than one can be peeled in the phase one if his neighbors except at most one are peeled before him. Afterwards we perform the phase two by deleting vertices of degree at most two and assigning them peel numbers two. In the phase i we delete vertices with degree i or less and assign them peel numbers i. We finish the process when there is no vertex left in the graph.

Similarly to degree sequence, a *peel sequence of the graph G* is a sequence of peel numbers of its vertices.

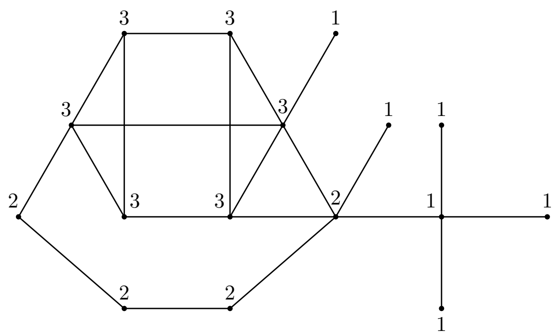

Figure 1: A graph with its peel numbers

It is easy to observe that the peel number of a vertex is at most its degree.

The peel numbers were already studied as an important property of large networks such as the Internet. They are often called *coreness* in the literature, e.g. [1, 2, 3]. A *k-core* is defined as an induced subgraph on vertices with peel number (coreness) at least k. Similarly a *k-shell* is an induced subgraph on vertices with peel number exactly k.

The algorithm computing peel sequence of a graph can be straightforwardly obtained from the definition. Efficient implementation is described in [1].

1.1 Peel Sequence Properties

Lemma 1.1. *Let $p_1, p_2, \dots p_n$ be a sequence of non-negative integers such that $p_1 \leq p_2 \leq \dots \leq p_n$ and let l be a number of integers with maximum value in the sequence ($p_n = p_{n-1} = \dots = p_{n-l+1} > p_{n-l}$). Then $(p_n)_n$ is a peel sequence of a graph if and only if $l \geq p_n + 1$.*

Proof. Let $p_1, p_2, \dots p_n$ be a sequence with $l \leq p_n$ and consider a graph G with such peel sequence. If we want to count its peel sequence we first have to peel (delete) vertices with peel numbers less than p_n first. After we delete them, there are $l \leq p_n$ vertices with peel number p_n. So every vertex has degree at least p_n. Since there is not a sufficient number of vertices we get a contradiction.

For the reverse implication we construct a graph G_p for an arbitrary given sequence with $l \geq p_n + 1$. First we take $p_n + 1$ vertices with maximum peel number and make the clique K_{p_n+1}. Then for each integer p_i in the sequence except the $p_n + 1$ integers already used we create a vertex and connect it to p_i vertices in K_{p_n+1}. Note that this can be done in arbitrary order. See Figure 2 for an example of such graph.

To finish the proof we observe that G_p has the peel sequence $(p_n)_n$. Every vertex outside of K_{p_n+1} corresponds to p_i in the original peel sequence and it has the same degree, thus its peel number is at most p_i. Moreover it is adjacent only to vertices in K_{p_n+1}, thus it can removed from the graph before any vertex from K_{p_n+1} during the computation of the peel sequence. Hence its peel number is p_i. The vertices in K_{p_n+1} have degree at least p_n and they can be removed after every other vertex, thus they all have the peel number p_n. □

The constructed graph G_p for a sequence $(p_n)_n$ is used later in this article.

1.2 Preserving Peel Sequence

Now we consider the following problem. Given the graphs G_1 and G_2 with the same peel sequence $p_1, p_2, \dots p_n$, we want to find operations that reconstruct a given graph G_1 into another given graph G_2 without changing

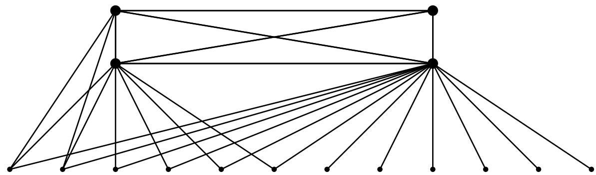

Figure 2: The graph G_p with the same peel sequence as the graph on Figure 1. The vertices in the clique K_{p_n+1} are highlighted.

the peel sequence during the process. This problem is similar to finding operations that preserve the degree sequence and that make from a graph any other graph with the same degree sequence. For the solution see [6, 5].

Moreover the operations should be local, i.e. they affect only a neighborhood of a constant number of vertices to a certain constant distance. We show that such operations exist for peel numbers and that they affect neighborhood of one vertex to distance two.

We want to make G_2 from G_1 by a set of operations and we do it through an intermediate graph G_p with the same peel sequence, i.e. from G_1 we construct G_p and from G_p we construct G_2 while always preserving peel sequence. We deal only with constructing G_p from a given G_1, since if we are able to make G_p from G_2, then we create G_2 from G_p with the reverse operations in the reverse order.

Theorem 1.2. *Given a graph G_1 we can construct the graph G_p with the same peel sequence as G_1 by creating graphs $G^0, G^1, G^2, \dots G^n = G_p$ such that G^i and G^{i+1} differs only by adding and deleting edges in the neighborhood of one vertex to distance two. The number of edges added or deleted by a single operation depends only on the peel number of a vertex. The peel sequence of all graphs G^i is the same as the peel sequence of G_1.*

Proof. Let $p_1, p_2, \dots p_n$ be the peel sequence of the graph G_1 and let k be the maximum number of this sequence. By Lemma 1 the number of vertices with peel number k is at least $k+1$.

Consider a computation of a peel sequence of G_1 by deleting vertices in the order $v_1, v_2, \dots v_n$. We show that the computation can be done in the same order (during the process) for all graphs G^i (but it can change if we want to create G_2 from G_p).

We fix the vertices $v_{n-k}, v_{n-k+1} \dots v_n$ and call them *fixed vertices*. First we make a clique from these fixed vertices, i.e. only add some edges, and create the graph G^0. This is necessary because otherwise the peel number of a fixed vertex can decrease later. We can split the creation of the clique into many operations, e.g. one operation would be adding an edge.

Since the vertices in the clique are the last ones to be deleted in the computation of the peel sequence, all other vertices are deleted before them. If there are only fixed vertices, then their degree is exactly k in G^0 and thus their peel number is k. Since the peel numbers of other vertices also do not change, by creating the clique the peel sequence remains the same.

The operation that makes G^i from G^{i-1} is rewiring one vertex v_i (which is not fixed) to the fixed vertices and adding an edge from the original neighbors of v in G^{i-1} to a fixed vertex if needed. We do this for the vertices $v_1, v_2, \ldots v_{n-k-1}$ creating the graphs $G^1, G^2, \ldots$.

More precisely when we want to rewire a vertex v_i with the peel number p_i we first delete all its edges and add p_i edges to the fixed vertices $v_{n-p_i+1}, \ldots\ v_n$. It may happen that some neighbor v_j of v_i with the peel number p_j has degree less than p_j (note that if so, p_j must be equal to p_i). In such case we add an edge from v_j to a fixed vertex that is not adjacent to v_j (there is at least one such fixed vertex).

Observe, that we delete exactly p_i edges, since we rewire in order of the peel sequence computation and in every step the peel number of any vertex is at most its degree (otherwise the peel number of v_i would decrease during the process). We add at most $2 \cdot p_i$ edges, thus the number of edges added or deleted during a single operation depends only on the peel number of the vertex v_i. Note that if the vertex v_i is already adjacent to the fixed vertices we do not need to delete and add again those edges.

It remains to show that the peel sequence is preserved during the process. We already observed this for creating the clique from fixed vertices and we want to prove the same for rewiring operation on vertex v_i ($i = 1, 2, \ldots\ n - k - 1$).

Since the degree of v_i is p_i and its neighbors in G^i are fixed vertices, the peel number of v_i does not change. If the neighbor v_j of v_i in G^{i-1} has the same peel number ($p_j = p_i$), we might add an edge to a fixed vertex, but it does not decrease or increase peel number of v_j – adding an edge is done only when the peel number would decrease. If we did not add an edge, the peel number remains the same, because in the computation of the peel sequence we delete v_i before v_j and v_j has degree at least its peel number. In the case $p_j > p_i$, the peel number of v_j stays the same, since the peel number of v_j cannot be decreased by deleting an edge to a vertex with strictly lesser peel number.

The peel numbers of the fixed vertices also do not change, since in the computation they are deleted after v_i. For other vertices, that are not v_i, adjacent to v_i or fixed, their peel number cannot be changed, because the edges incident to them were not deleted or added and peel numbers of their neighbors also did not change.

□

1.3 Example of creating G_p

We give an example of the process described above in which we make graph G_p on Figure 2 from a graph on Figure 1. We show the graphs G^0, G^1, G^2, G^6, G^7 and $G^{15} = G_p$ which are made during the process.

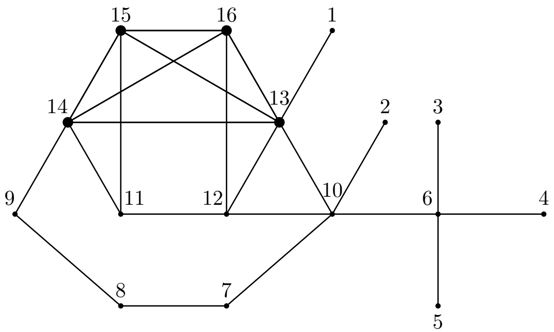

Figure 3: Graph G^0 made from the graph on Figure 1 by creating the clique from the fixed vertices (highlighted). The numbers mark the order of computation of the peel sequence.

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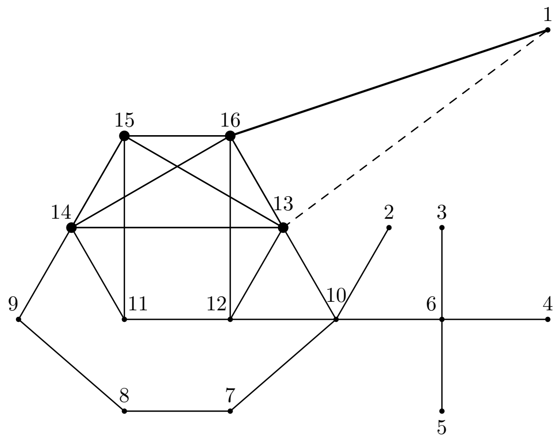

Figure 4: Graph G^1. Deleted edge is dashed and the new added one is bold.

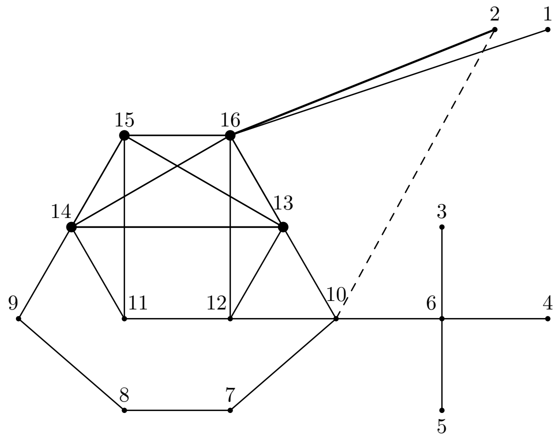

Figure 5: Graph G^2.

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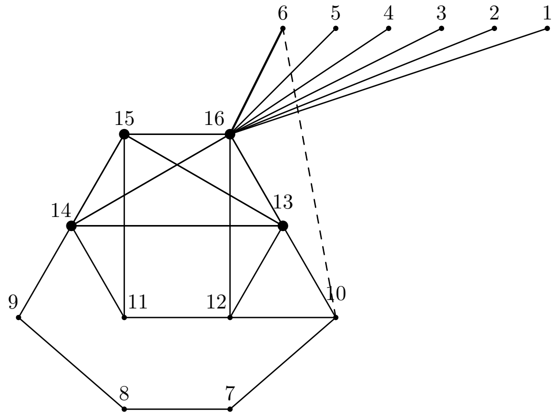

Figure 6: Graph G^6 with all vertices with peel number 1 rewired.

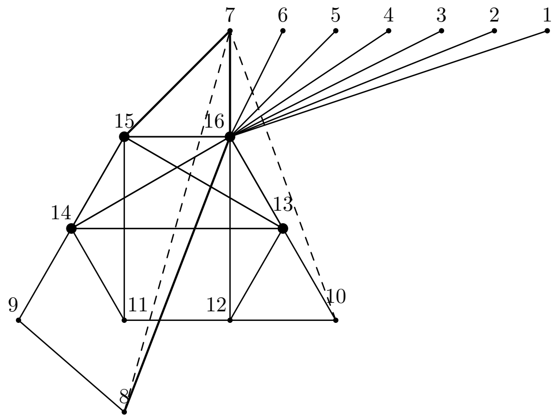

Figure 7: Graph G^7. Note that we have to add an edge between the vertices 8 and 16.

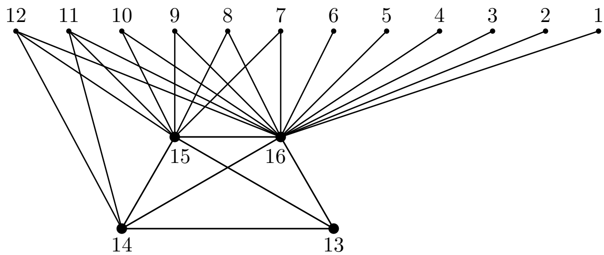

Figure 8: Graph $G^{15} = G_p$ is the result of the process.

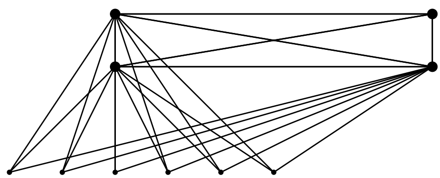

Figure 9: The graph G_p for a 3-regular graph on 8 vertices.

A Trip to Prague: What I Have Learned

Marissa Loving
University of Hawaii - Hilo

These past days in the Czech Republic have been some of the busiest and most exciting in my life. From morning seminars to afternoon romps through castles and gardens to evenings spent absorbing the local food and culture, not a minute has gone to waste. As an aspiring mathematician it has been a great honor to meet and speak to mathematicians of such excellence and importance as Jaroslav Nešetřil, Pavel Valtr, Jiří Fiala, and many others.

I really enjoyed our first week of lectures and problem solving. Although I found each of the lectures stimulating, I was particularly excited by the lecture given by Yared Nigussie on W.Q.O. (Well-Quasi Ordering) Theory. I had never found myself intrigued by the topics I had encountered in order theory, but this talk gave me a new perspective.

Yared began by introducing us to the concept of a quasi-order. The idea of a quasi-order is similar to that of a partial order, which I was familiar with from some of my computer science classes. Much like a partial order can be thought of as a more relaxed form of a total order, that is, it allows us to have some notion of *greater than* or *less than* in a set where not necessarily all elements are related to each other, a quasi-order can be thought of, intuitively, as some more relaxed or general form of a partial order.

Definition: A <u>quasi order</u> $(Q, \leq)$ consists of a set Q and a relation $\leq$ such that $\leq$ is both reflexive and transitive.

Thus, even the definition is very similar to that of a partial order except it omits the need for $\leq$ to be antisymmetric. Note that, given this definition. It is possible for two distinct elements, $q, q' \in Q$ to have the property that $q \leq q'$ and $q' \leq q$.

With this mathematical machinery in place, Yared returned to the familiar concept of the Well-Ordering Principle with this question in mind, "Does a similar concept hold for quasi-ordered sets?" It turns out that there are some pretty neat results proved about Well-Quasi Ordered sets. However, before we were able to discuss some of these results we needed to develop a notion of what it means to be not just quasi-ordered, but *well* quasi-ordered.

Definition: A quasi-order $(Q, \leq)$ is said to be well quasi-ordered if for every infinite sequence $q_1, q_2, q_3, \ldots$ you will always find $i < j$ such that $q_i \leq q_j$.

So what is this *similar concept* that was mentioned? We consider Higman's Lemma as well as a theorem by Kruskal. The versions introduced to us by Yared are given below.

Higman: If Q is well quasi-ordered, then $Q^{<\omega}$ is also well quasi-ordered under "embedding".

Kruskal: Trees under embedding are well quasi-ordered.

Although the original proof of Higman's Lemma was very long and complicated, Nash Williams gave a very simple and elegant proof of this using an argument known as the minimal-bad-sequence argument. The proof uses three infinite "tapes", a constructive approach, and a series of contradictions to arrive at the desired conclusion. Yared went over Nash Williams' proof with us and I was excited to be able to follow along quite well.

This was not always the case during the Midsummer Combinatorics Workshop the following week because of the select audience the talks were often given at such a high level that I found myself quite lost after the first few definitions. Nonetheless, it was wonderful to be immersed in such an intense mathematical environment and I found myself eagerly listening to talks even when I had very little of the mathematical mechanics to understand the more intricate details of the concepts and proofs being presented.

Several of the talks which stood out to me were Dhruv Mubayi's talk, *Quasi-random Hypergraphs*, Tomáš Gavenčiak's presentation, *Hyper-tree-depth and Hypergraph Pairs*, and Kolja Knauer's discussion, *Simple Tree-width*. However, I think I most enjoyed Jan Hubička's talk entitled *Order of Locally Constrained Homomorphisms*. This may be attributable, in part, to the fact that I actually knew much of the vocabulary that was being employed. I also thought it was really neat to see the work of Nešetřil being cited in his own workshop.

Jan's presentation began with an introduction to the set, G, of graph homomorphisms on a graph with the identity homomorphism as a relation $\leq$ form a quasi-order $(G, \leq)$. If we consider to such sets G and H we can say that G and H are hom-equivalent if and only if $G \leq H \leq G$. Now, what if we consider the set of functions that are only locally surjective, injective, or bijective, respectively? It turns out that these locally injective, surjective, and bijective homomorphism induce a partial ordering and we can, in fact, prove the following lemma.

Lemma: Every locally surjective or bijective homomorphism $F : G \to H$ is surjective when H is connected.

We can use the following result by Nešetřil

Theorem: Every locally injective homomorphism $F : G \to G$ is an automorphism of G for connected graphs G.

to prove that if $drm(G) = drm(H)$, then every locally injective or surjective homomorphism is locally bijective for connected graphs G where drm is the degree refinement matrix.

So, in some sense, it seems that the degree refinement matrix describes or at least distinguishes the equivalence classes of the equi-parititions of our graph.

Jan finished by noting that the homomorphism order is universal on partial orders and lattices and asking the audience to consider the case of duality pairs. This was how many of the speakers ended their talks, not with this specific question, but simply with a series of questions, conjectures, or open problems, which lent itself to the overall flexible, hands-on feeling of the workshop. An environment that I found very exciting because it helped me envision myself in a similar setting as an actual mathematician and researcher.

Although I was not able to work on any of the questions that were posed during the Midsummer Combinatorics Workshop, the speakers during the first week of our stay in Prague did provide all of us students with some problems to tackle, which were quite interesting as they involved topics, such as graph theory, that I had never worked out proofs for.

Problem Set

1. For every natural number $n \geq 2$ there is a cograph of order n that is not an equivalence graph.

> *Proof:* Consider the complete graph of order n for any $n \in \mathbb{N}$, that is K_n. First note that K_n is a cograph for every n. Now, suppose we delete a single edge of K_n, to form a graph we will call K_n'. K_n' is still a cograph. However, K_n' is no longer an equivalence graph because their exist two vertices of K_n' that are connected by a path, yet no edge exists between them. Hence, for every $n \in \mathbb{N}$ there exists a cograph of order n that is not an equivalence graph.

2. A graph is perfect if the chromatic number of each induced subgraph equals the clique number of the induced subgraph.

(a) Give an example of a graph that is not perfect.

Any odd numbered cycle.

(b) Show that all equivalence graphs are perfect.

Proof: Let G be an equivalence graph of n vertices. Note that every induced subgraph of an equivalence graph is also an equivalence graph. Thus, the chromatic number of every induced subgraph will be equal to the clique number of that subgraph. Namely, a subgraph of order m will also have chromatic and clique number m since it is an equivalence graph. Hence, G is perfect and we can conclude that every equivalence graph is perfect.

(c) Show that the union of two vertex-disjoint perfect graphs is perfect.

Proof: Let G and H be two vertex-disjoint perfect graphs. Consider an induced subgraph I of $G \cup H$. We have three cases: all the vertices of I are contained in G, all the vertices of I are contained in H, or the vertices of I are contained in both G and H.

If all the vertices of I are contained in G, then I is an induced subgraph of the perfect graph G and so its clique number is equal to its chromatic number, trivially. Without loss of generality, this argument can be applied to I when all of its vertices are contained in H.

If the vertices of I are contained in both G and H and these are vertex-disjoint graphs then the vertices of I contained in G and those contained in H are completely disjoint sets. Thus, I is the vertex-disjoint union of two induced subgraphs, one of G and one of H. Because G and H are vertex disjoint it is clear that the clique number of I will be equal to the maximum of the clique numbers of the two subgraphs induced in G and H, respectively, by I. Similarly, the chromatic number of I will be equal to the maximum of the chromatic numbers of the two subgraphs induced in G and H, respectively, by I. Since both G and H are perfect graphs, their induced subgraphs will each have clique number equal to chromatic number and we can conclude that the clique number of I will be equal to the chromatic number of I. Hence, the vertex-disjoint union of two equivalence graphs and thus, an equivalence graph.

Hence, in any case, every subgraph I of $G \cup H$ is an equivalence graph and we can conclude that the vertex-disjoint union of perfect graphs is perfect.

3. Determine the tree width of all outer planar graphs.

   Proposition: The tree width of any outer planar graph is at most 2.

   Proof: Let G be an outer planar graph. Consider the graph G' created by adding edges to G until we have attained a maximal outer planar graph. Consider the reduced dual, T, of G', that is the dual of G' minus the vertex for the outer plane. We claim that T is a tree decomposition of G'.

   First, we will show that T is a tree by showing that it does not contain a cycle. Suppose to the contrary that T contains a cycle. If T contains a cycle of length n then there exists at least one vertex in G' that is completely bounded by faces of G'. However, G' is an outer planar graph and therefore cannot contain an internal vertex, a contradiction. Thus, we conclude that T is indeed a tree.

   Second, we will show that T is indeed a tree decomposition of G' because:

   – every node of T is a subset of V'_G, since every node of T consists of the edges and vertices bounding a plane region of G';

– by definition every edge of G' is contained in a node of T;

– and we know that every $v \in V_G$ is contained in at least a single node of T and, additionally, that if v appears in multiple nodes of T these nodes will be adjacent, by construction, and we can conclude for every $v \in V_G'$ the nodes containing v will induce nonempty and connected subtrees in T.

So T has width 2 because $f(t) = 3$ for ever $t \in V_T$. Thus, we have found a tree decomposition of G' with width 2 and can conclude that the tree width of G' is at most 2. Because we can obtain our graph G by simply deleting edges from G' we know that G is a minor of G'. Thus, the tree width of G' forms an upper bound on the tree width of G. Hence, the tree width of any outer planar graph is at most 2.

4. Given graph G, the girth of G, denoted $g(G)$, is the number of edges in the smallest induced cycle in G. If G is a cograph, what is the largest possible value of $g(G)$? What is the smallest?

Proof: Consider a cograph G. Note that, for every n P_n is a subgraph of C_n. In fact, it is clear that for every n $P_1, P_2, \ldots, P_{n-1}$ are induced subgraphs of C_n. Thus, since G is a cograph and P_4 free we can conclude that it does not contain an induced subgraph of C_n for $n \geq 5$. Hence, $g(G)$ is at most 4. Trivially, consider the smallest cycle, C_3. Note that C_3 is indeed a cograph with girth 3. Hence, it is clear that $g(G)$ is no less than 3 for any cograph G.

5. Consider the set of bipartite graphs. Is it true that every bipartite graph is a cograph? What if we restrict ourselves to the set of complete bipartite graphs?

Proof: Note that every cycle of even length is a bipartite graph. However, C_6 contains an induced P_4.

Suppose we look instead at only complete bipartite graphs. Consider a complete bipartite graph G that is formed by joining the vertex disjoint, empty graphs U and W and a collection ω of 4 vertices in V_G. We have three cases, either $\omega \subset V_U$, $\omega \subset V_W$, or $\omega \subset V_U \cup V_W$.

If $\omega \subset V_U$, then the subgraph induced by ω is an empty graph of 4 vertices. Without loss of generality the same holds for $\omega \subset V_W$.

If $\omega \subset V_U \cup V_W$, then we have four cases, but these can be reduced to two without loss of generality. Note that either one vertex in ω lies in U and three in V or two vertices lie in both U and V.

In the first case, since G is a complete bipartite graph, then we have the induced subgraph $K_{1,3}$ which does not contain P_4 as an induced subgraph.

Similarly, in the second case, since G is a complete bipartite graph, then we have the induced subgraph $K_{2,2}$, which once again does not contain P_4 as an induced subgraph.

Therefore, there is no collection of 4 vertices that induce P_4 and we conclude that every complete bipartite graph is indeed a cograph.

DIMACS REU 2012 - Prague Report

Theresa Lye
Rutgers University

The following report contains a discussion on two talks, one from the REU seminar series and one from the MCW, and solutions to several problems.

Orthogonal One-Bend Drawings, Pavel Valtr

Orthogonal one-bend drawings are a way of depicting graphs in which all edges must be straight lines running in one of the cardinal directions (completely horizontal or vertical running parallel to the x or y axis, no diagonal edges are allowed) and are only allowed to bend once in a 90 degree angle per edge. The edges may cross over one another, but may not overlap in such a way that two edges would be indistinguishable from one another. Not all graphs can be drawn as an orthogonal one-bend drawing, and we analyze the conditions necessary for a graph to be able to be depicted as an orthogonal one-bend drawing. We first consider two dimensional space.

The first observation is that all vertices must have a degree less than or equal to four. This is a necessary condition condition because if a vertex has a degree greater than four, there is no way to draw the edges emerging from the vertex using only the four cardinal directions.

Note that if all vertices had degree four, then the number of edges $|E|$ in the graph would be equal to $2|V|$. There would be four edges per vertex, but since each edge is shared by two vertices, $|E| = \frac{4}{2}|V| = 2|V|$.

With only the first observation, this implies that the maximum number of edges for a graph with an orthogonal one-bend drawing is $2|V|$. However, this is not true. The maximum number of edges is actually $2|V|-2$. This is because the vertices at the leftmost, rightmost, topmost, and bottom-most portions of the graph cannot have an edge going further in the respective directions (leftmost vertex cannot have an edge traveling left, and so on). Therefore, we lose 4 degrees, which means two less edges. Thus, if a graph has an orthogonal one-bend drawing, then $|E| \leq 2|V| - 2$. We have been able to draw an orthogonal one-bend drawing that has $|E| = 2|V| - 2$.

We would then like to extend this to three dimensional space. There are now six directions in which edges can move, but all previous rules still apply. Extending the above arguments in a simple manner, each vertex can have a maximum degree of six, while the number of edges is bounded as $|E| \leq 3|V| - 3$. We have made a few attempts at drawing an orthogonal one-bend drawing in three dimensions with $|E| = 3|V| - 3$ for low $|V|$, but so far have been unsuccessful. Whether we are simply having difficulty in visualizing an orthogonal one-bend graph in three dimensions or if there is a tighter bound on the number of edges in an orthogonal one-bend graph in three dimensions remains to be proven. It may be possible there are more conditions to consider: for instance, it seems impossible to connect vertices not within the same plane because it requires more than one bend in an edge.

Twins in Sequences, Maria Axenovich, Yury Person, and Svetlana Puzynina

A twin consists of two large disjoint substructures that have the same parameters. In the case of sequences, a twin could be two identical subsequences. The subsequences need not be continuous. The characters that comprise the subsequence may be scattered throughout the sequence.

A primary question discussed at the conference was how large the twins could be in any given binary sequence. It was argued that any binary sequence of length n could be split into two identical subwords (twins) about half the length of the sequence with a short remaining subword.

This could be done by partitioning the sequence into ϵ-regular partitions, except for perhaps a few. Let S be the sequence, $|S|_q$ be the count of the character q in the sequence, and d_q be the density of the character q in the sequence, defined as $d_q = \frac{|S|_q}{|S|}$. An ϵ-regular sequence applies a restriction on the density of the characters within the sequence.

Once a sequence has been partitioned into mostly ϵ-regular partitions, one can select repeated characters from each partition to construct the two identical subsequences such that the subsequences are twins consisting of about half the sequence each, with only a few characters left over that are not included in the twins.

Problems

Show that G contains an induced 4-path if and only if the complement of G contains an induced 4-path.

The complement of a 4-path (P_4) is also a 4-path, that is $P_4 = \overline{P_4}$. Therefore, if graph G contains an induced P_4, then its complement $\overline{G}$ will contain $\overline{P_4}$, which equals P_4. If $\overline{G}$ contains P_4, then its complement G will contain $\overline{P_4} = P_4$.

A graph is perfect if the chromatic number of each induced subgraph equals the clique number of the induced subgraph. Show that all equivalence graphs are perfect.

Equivalence graphs consist of all complete graphs and disjoint unions of complete graphs. First, consider that all complete graphs must be perfect graphs. Because all pair of vertices are connected, each induced subgraph will consist of one clique that contains all of the vertices in the induced subgraph. Thus, the clique number of the induced subgraph will equal the number of vertices in the induced subgraph. Also, because all vertices are connected, each vertex in an induced subgraph must be assigned a different color. Thus, the chromatic number of each induced subgraph must also equal the number of vertices in the induced subgraph. Thus, all complete graphs are perfect.

If the equivalence graph consists of a union of complete graphs (let us call the individual complete graphs in the union as components of the equivalence graph), then it must still be perfect. If an induced subgraph of the equivalence graph consists of vertices/edges from only one component, then naturally the induced subgraph must have equal clique and chromatic number because the induced subgraph was induced from one complete graph, which must be perfect. If an induced subgraph of the equivalence graph consists of vertices/edges from multiple components (let us call the set of vertices/edges induced from one of the components as an induced component), then the clique and chromatic number of the induced subgraph will be maximum clique and chromatic number among the induced components. Because the components have no edges between them, the induced subgraph will essentially consist of a union of cliques, with the largest clique determining the clique and chromatic number for the entire subgraph. Because each induced component was induced from a perfect graph, the maximum clique and chromatic number will be equal. Thus all induced subgraphs will have equal clique and chromatic number, all equivalence graphs are perfect.

Show that the union of two vertex-disjoint perfect graphs is perfect.

Let G be a perfect graph, and H be another perfect graph. Because the two graphs are vertex-disjoint, there are no edges going between G and H in their union, $G \cup H$. We must show that all induced subgraphs of their union have their chromatic number equaling their clique number.

If an induced subgraph from the union includes only vertices and edges from either G or H, then the induced subgraph has equal clique and chromatic number because G and H individually are already perfect graphs.

But consider if the induced subgraph from the union includes vertices and edges from both G and H. In this case, there will be two disjoint sets of vertices, one that is induced from G and one that is induced from H, which we will call G' and H' respectively. Because there are no edges going between G' and H', the clique number of the entire induced subgraph must be the maximum between the clique numbers of G' and H'. Also because G' and H' are disjoint, the maximum chromatic number between the chromatic numbers of G' and H' must also be the chromatic number of the entire induced subgraph. Because G' and H' are induced from perfect graphs, their chromatic numbers must equal their clique numbers. Thus, the induced subgraph will have equal chromatic and clique numbers, which will be the maximum chromatic and clique numbers between G' and H'.

Is every bipartite graph a cograph? Is every complete bipartite graph a cograph?

No, not every bipartite graph is a cograph. A path on four vertices itself is a bipartite graph, and is clearly not a cograph.

Yes, every complete bipartite graph is a cograph. Assume the complete bipartite graph G has a path of four vertices, P_4. Then it must have edges $(v1, v2)$, $(v2, v3)$, and $(v3, v4)$. By definition of a bipartite graph, the vertices can be split into two sets U and V. We can say that $v1$ and $v3$ are contained in U, while $v2$ and $v4$ are contained in V. By definition of a complete bipartite graph, the graph must also contain the edge $(v1, v4)$. This will cause G to no longer have a P_4. Thus, a complete bipartite graph cannot have a P_4 and must be a cograph.

DIMATIA/MCW Program Report

Michael Poplavski
University of Central Florida

1 Introduction

Our trip included talks in the morning and afternoon the first week by faculty at Charles University on Wednesday through Friday. The second week included an exciting set of lectures in the Mid-Summer Combinatorial Workshop. Our last week consisted of talks on Monday and Tuesday, then our departure back the U.S. on Wednesday. During our trip we were accompanied by a set of local Czech students who were kind enough to prepare exciting tours and adventures for us.

2 Workshop Talks

Our first talk and the one I found most interesting was by Zdeněk Dvořák. The talk was mainly on tree width where he discussed specific properties and presented some interesting problems to us. First in order to understand what treewidth is we were introduced to what is known as tree decomposition. Tree decomposition is where you have an optimal mapping where each vertex corresponds to a set of vertices of the graph. Another area where tree decomposition is proven to be an important area to study is string processing and spell checking. In this area the string is broken up into vertices and the prefix of a single word may have multiple connections to other words. The treewidth of a graph is the minimum width of a graph G of all possible tree decompositions of that graph G. Let us note that the width of the graph is simply the size of the largest set of vertices of that graph minus one. I will now discuss some of the interesting properties of treewidth that we discussed during the lecture: One property to note is that if G is already a tree $\text{tw}(G)$ is less than or equal to one. This is a fairly intuitive property to prove. A possible idea for this proof would be induction on the tree where you remove a leaf of the tree, until you are left with a set of vertices of size less than three. Next the treewidth of a complete graph with n vertices is equal to n minus 1. This property as well seems relatively intuitive. One would partition the graph into sets of vertices where each set contains n vertices making the treewidth simply n minus 1. Let us also note that if you contract the edges of a graph it does not increase the treewidth. I will now go on to discuss two interesting problems that he gave us to solve. Our first problem was on the treewidth of an outer-planar graph. Firstly, an outer-planar graph is graph that can be drawn with no edge crossings and where no vertex is completely surrounded by edges. This can be done by making the graph a maximal outerplanar graph. At this point the graph contains triangles as faces which is a minor graph of the other, and by definition this will then make the treewidth at most 2. The second problem was to show that the treewidth of a graph G is less than the smallest degree of the graph. This problem is best thought of as a tree where your proof is based off induction on the set of vertices of that tree. There are the base cases when your graph has one and two vertices. One vertex has a tree width of zero and a smallest degree of zero. Two vertices connected has a tree width of one and a smallest degree of one. Now you can say there exists a set of vertices that are connected to another set in the tree beginning from one of the leaf vertices because there must exist at least one edge. So the treewidth would always be less than or equal to the smallest degree because of the vertex sets overlapping.

3 Midsummer Combinatorial Workshop Talks: Binary Paint Shop Problem

We are given a word with n characters of length $2n$ where every character occurs exactly twice. The objective of the problem is to color the letters of the words using two colors, such that each letter receives both colors and the number of color changes of consecutive letters is minimized. A step by step example of the greedy algorithm I came up with below:

Word = ABBCDCDA

Step 1: Color AB blue

Step 2: Color BCD red

Step 3: Color CD blue

Step 4: Color A red

Coloring in 4 changes = $\underline{\text{AB}}$BCD$\underline{\text{CD}}$A, where blue is underlined and red is not. In this example the number of color changes was three with n equaling 4. In the talk they proved a new bound on the expected number of changes of the number of color changes averaged across the permutations of the $2n$ letters of the greedy algorithm from at most $2n/3$ to $2n/5$.

DIMATIA/MCW Program Report

Ixtli-Nitzin Sanchez
San Jose State University

1 Introduction

My time in Prague while exposing me to a unique culture has also opened and extended my original view of the mathematical world. After attending the REU and MCW talks I feel privileged to have been invited and allowed to attend the conference sessions that DIMATIA has pre-arranged for myself and other my fellow REU student colleagues.

2 REU Talks

During the first week in Prague there were several graph theory workshops created for the five REU students. Professors both from Charles University and universities from the United States gave lectures on a variety of topics some of which included graph theory, tree width, and WQO (Well Quasi Ordering) theory. While all of them where all interesting and insightful the two talks given by Jiří Fiala and Pavel Valtr intrigued me the most.

Jiří Fiala, a professor from Charles University, spoke about an abstract strategy board game called Ypsilon that was created in the 1950s. When playing Ypsilon there are two ways to win. The first method is that the person who connects all three sides of the board (assuming that the two players are playing on triangular board) before the other player wins. The second way to win is to connect a line from a corner of the triangle to the opposite side.

After Fiala described the game and its rules he then posed two questions. The first question was, "Can this game end up in a draw?" Which the answer was discovered to be that the game can never end up in a draw and there is always a winner. The second question was, "Do any of the two players have a winning strategy?" I only figured out the answer to this question after playing several games against my other REU colleagues. It turns that for the game Ypsilon the first player always has the advantage and unless the first player makes a mistake he will usually win against his opponent.

Pavel Valtr, also a professor from Charles University, spoke about graphs that have orthogonal one-bend drawings. Valtr gave several examples while in class and because he explained clearly they were easy to comprehend. He also asked the REU students to solve several problems during his lecture and he always provided enough time to solve each question.

3 Problems solved

3.1 Pavel Valtr Exercise 1

Find "many" graphs with $|E| \leq 3|V| - 3$ admitting a nice drawing in $\mathbb{R}^3$ for K_7.

The final solution and design that provided the most amount of edges can be seen below in Figure 1. The maximum amount of edges for this graph is 14. I tried several different schemes, but every attempt was always less then 14 edges. I also noticed that if one wanted to connect two vertices that were on parallel planes then the rule of one-sided bend only would be broken because the person constructing the graph would have to make two bends in order to connect the pair.

Also for the equation $|E| \leq 3|V| - 3$, it was never possible for a given number of vertices to reach the theoretical maximum number edges, denoted by the letter E, assuming that edges can only be made in the (x, y, z) direction.

3.2 John Gimbel Exercise 1

Show that G contains an induced 4-path if and only if the complement of G contains an induced 4-path.

See Figure 2.

4 MCW Talks

I was amazed that I was able to attend a conference while in Prague with mathematicians from all over the world who have specialized in their particular field. A group photo taken while visiting the Strahov Library and a copy of the photograph was given to each individual the following day while at the MCW conference. The picture reminded me of group photo taken of

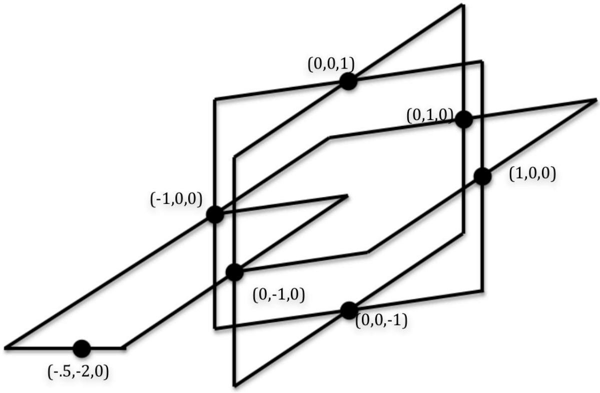

Figure 1: 3D construction for K_7, maximum number of edges is 14.

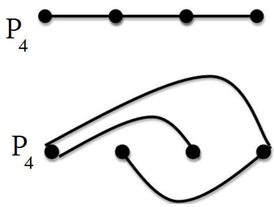

Figure 2

the physicists and chemists present during the 1927 Fifth Solvay International Conference on Electrons and Protons. I enjoyed looking at the group photo taken at the Strahov Library because I, an undergraduate with no publications and no specialization in mathematics, was in the picture with these well-known mathematicians. I was fortunate to be chosen and I am thankful the opportunity.

The talk I found most interesting was by Jakub Mareček and it was titled "Data Structures for Stochastic Scheduling with Applications in GPGPUs". The research that Mareček spoke about related to the summer REU research that I conducted while at DIMACS. The research I conducted this summer was called "Using New York Cities 2009 historical traffic data to develop an iPhone routing application". While my research involved a number of aspects the subclass within my research on ride sharing was applicable to Mareček's research on stochastic scheduling. The purpose of stochastic scheduling is to find a preemptive or non-preemptive scheduling policy, for choosing which job to serve at each decision epoch and concerned with various optimization criteria.

Mareček's talk first discussed the motivation and the background for his work in which was broken down between ARM, GPGPUs, stochastic scheduling with presidencies. His research work has been a joint effort between ARM and the University of Nottingham. ARM is a processor architecture and the company designs many low-cost and power-efficient RISC(reduced instruction set computing) processors used on cell phones today. The background for the research that he is conducting is that he uses an ARM processor with a GPGPUs (general-purpose computing on a graphics processing unit) called OpenCL, which is language for writing "kernels" and controlling application programming interfaces.

Mareček then uses stochastic scheduling to create a weighted system for policy π:

$$J(\pi) = \limsup_{t\to\infty} \frac{1}{t} \sum_{q\in Q} w(q)\mathbb{E}[a_q^\pi(t)]$$

Where a_q is the number of jobs from queue completed by time t.

For the ride share program there would be a certain amount of hosts, the people driving, and riders, the people seeking to carpool. Implementing stochastic scheduling into my ride share program would allow me to specify certain variables and then initiate a preemptive scheduling policy, for choosing which rider to serve at each decision. Some of the variables may include the proximity of where the rider is in relation to the host, time of travel from origin to destination, and other distance related variables that can be included.

5 Additional Commentary

There were several presentations that were too specialized to follow without the specific mathematical background. If a schedule was provided to the REU students or at least of the mathematicians, that were attending the conference, before the beginning of the conference then it would have given the REU students ample amount of time to look up and research the mathematicians. The talks nonetheless were interesting to listen to and provided insight into fields I had no previous knowledge of.

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[2] J. Nino-Mora. *Stochastic scheduling.* In Encyclopedia of Optimization, 2nd edition, C.A. Floudas and P.M. Pardalos, eds., pp. 3818–3824. Springer, New York. Retrieved from: `http://halweb.uc3m.es/jnino/eng/pubs/ssche.pdf`

DIMATIA/MCW Program Report

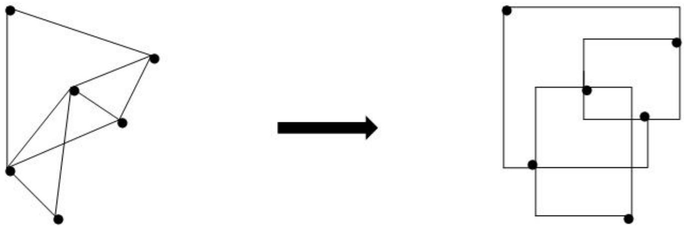
Ethan Schwartz
Emory University

When I was informed of my selection to the Rutgers DIMACS REU, and subsequently of my selection to participate in the international component of the REU, I was excited at the possibility of being introduced to high-level mathematics and research that would help me in my mathematical studies and my own personal research. As a pre-medical student studying applied mathematics at the undergraduate level, I saw the opportunity to study at the Charles University in Prague for two weeks, and attend the annual Midsummer Combinatorial Workshop as a great learning experience. It is not everyday that one may hear lectures from the leading mathematicians in their respective fields.

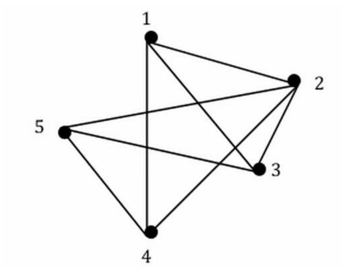
After arriving to Prague on Tuesday, the following three days were filled with lectures designed solely for the REU participants. These lectures ranged in topics: Zdeněk Dvořák spoke on the tree width of graphs; Pavel Valtr introduced us to orthogonal one-bend graphs; John Gimbel offered an overview of many terms in graph theory, but most specifically cographs; Yared Nigussie delivered an emphatic lecture on WQO (Well-Quasi-Ordering) Theory; and Jiří Fiala concluded the week by introducing the game Ypsilon, as well as its derivative Reverse-Ypsilon.

While all of these lectures were very interesting and contained new mathematics, and although Jiří Fiala's lax Friday talk revolving around the game of Ypsilon was quite engaging and different, I will focus on Pavel Valtr's lecture on orthogonal one-bend drawings. Throughout the summer at Rutgers, and in my time in Prague, I was introduced to a lot of new mathematics, including graph theory, combinatorics, and discrete mathematics. This talk, by Pavel Valtr, was most interesting to me because it built on the ideas of graph theory, and presented some ideas that I had not yet seen during the summer. It offered a completely new way of viewing the graphs that I had been seeing throughout my summer at Rutgers. To summarize the main ideas of this talk:

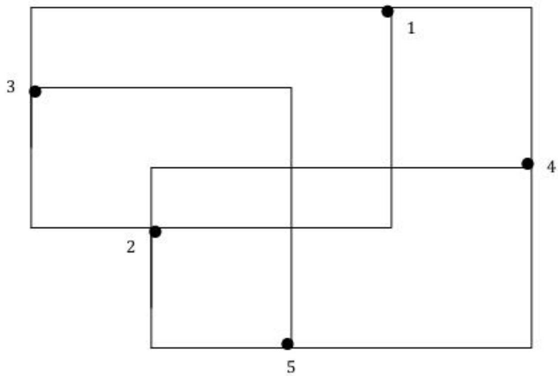
- If $G = (V, E)$ has an orthogonal one-bend drawing, then $|E| \leq 2|V| - 2$. Given a graph $G = (V, E)$, then the orthogonal one-bend drawing of G consists of vertices (points), and edges (one distinct horizontal + one distinct vertical component). Thus, from this definition, it can clearly be seen that if G has an orthogonal one-bend drawing, then every vertex of G has degree ≤ 4.



- If $G = (V, E)$ has an orthogonal one-bend drawing, then $|E| \leq 2|V| - 2$.
- At this point in the lecture, Pavel Valtr issued a challenge to the REU group. He drew the following graph on the board, and asked us to find its orthogonal one-bend drawing, assuming that it exists.



- After Valtr gave the group time to solve the issued challenge, I thought I had a solution to his challenge. He had me come up to the board to present my solution:



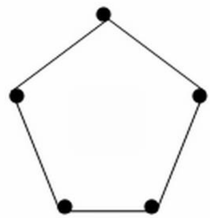
- Valtr continued to give a few more observations and theorems concerning the orthogonal one-bend drawings. It was also very interesting when he proposed completing the same type of drawings but in $\mathbb{R}^3$. Thus, in such drawings, edges would consist of one distinct component in the x-direction, one distinct component in the y-direction, and one distinct component in the z-direction.
- While much of mathematics is beautiful on its own, and does not need a clear-cut application, I was impressed with Valtr's impromptu reasoning when asked for an application of such orthogonal one-bend drawings. He said, for example, that it could be used to model parts of large modern cities, many of which are situated in a block format.

Following the first half-week of lectures, the subsequent week consisted of the Midsummer Combinatorial Workshop (MCW). Unfortunately, for most of the MCW, I was very ill, and thus could only attend a few talks at the beginning and at the end of the week. One talk in particular was very interesting to me, because the speaker was a graduate student at Emory University, at which I am an undergraduate student. Vindya Bhat spoke on research involving an improved upper-bound on the density of quasi-random hypergraphs. Bhats research extended the ErdősStone theorem to quasi-random hypergraphs, and attempted to improve the upper-bound on the density of such graphs. This talk was interesting to me because, as aforementioned, the speaker is a graduate student at my home institution. It opened my eyes to the possible mathematical research that I could participate in as an undergraduate student. Throughout our lectures before the MCW, the lecturers gave us numerous problems to attempt to solve. The problems mainly dealt with graph theory and combinatorics, and were of varying degrees of difficulty. Some of these solutions were attained while working with other students in the REU. The solutions follow:

1. Gimbel offered the following problem:

   A graph is perfect if the chromatic number of each induced subgraph equals the clique number of the induced subgraph.

   (a) Give an example of a graph that is not perfect.
   Solution:

   (b) Show that all equivalence graphs are perfect.
   Solution: Let $G = (V, E)$ be an equivalence graph of n vertices, denoted as K_n. It is important to note that every induced subgraph of K_n is a complete graph, due to the definition and nature of equivalence graphs. Thus, $\chi(G)$ for all induced subgraphs will equal $\omega(G)$ of that induced subgraph. In other words, an induced subgraph of order m will exhibit the property $\omega(G) = \chi(G) = m$, since it is K_m. Hence, this induced subgraph is perfect and we can conclude that every equivalence graph is perfect.

2. Gimbel also offered the following problem:

   Given graph G, the girth of G, denoted $g(G)$, is the number of edges in the smallest induced cycle in G. If G is a cograph, what is the largest possible value of $g(G)$? What is the smallest?

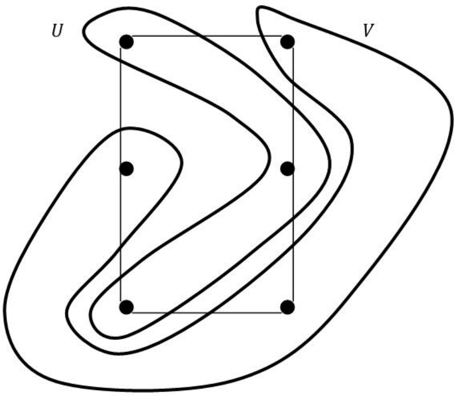
   Solution: Consider an arbitrary cograph $G = (V, E)$. By definition, G does not contain P_4 as an induced subgraph. Thus, C_4 can exist within this graph; however, any cycle larger than length 4 cannot exist within this graph. Also, consider any K_n, $n \geq 3$. In such cographs, the smallest induced cycle is C_3. Thus, the $g(G) \leq 4$ for a cograph.

Also, since C_2 or C_1 cannot exist within the simple graphs under consideration, $g(G) \geq 3$. Thus, by these properties $3 \leq g(G) \leq 4$ for a cograph.

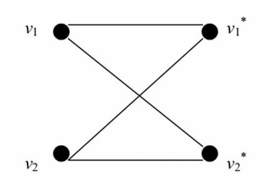
3. Gimbel also offered the following problem:

   Is every bipartite graph a cograph? Is every complete bipartite graph a cograph?

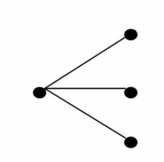
   Solution: No. Every bipartite graph is not a cograph. Consider, for example, C_6. C_6 is a bipartite graph, as shown below. However, C_6 contains P_4 as an induced subgraph. In fact, all C_n, where n is an even number ≥ 4, are examples of bipartite graphs that are not cographs.



   It can be shown that every complete bipartite graph is a cograph. By definition, a complete bipartite graph is a graph in which every vertex of the first set, U, is connected to every vertex of the second set, V. Thus, consider vertices v_1 and v_2 in set U, and vertices v_1^* and v_2^* in set V of a complete bipartite graph $G = (V, E)$. Since G is a complete bipartite graph, v_1 is connected to v_1^* and v_2^*, and v_2 is connected to v_1^* and v_2^*, but v_1 is not connected to v_2. Thus, the four vertices are connected as shown below. In other words, in such a graph, any induced subgraph will be $K_{2,2}$.



This graph shown above is a cograph, as there is no way to induce P_4. All complete bipartite graphs in which U contains two or more vertices, and V contains two or more vertices, has the subgraph shown above. Thus, any bipartite graph with two or more vertices in each disjoint set is a cograph. Now consider a complete bipartite graph in which U contains one vertex, while V contains three or more vertices. This graph will have the subgraph $K_{1,3}$, as shown below. This subgraph is also a cograph. Thus, any complete bipartite graph in which U contains one vertex and V contains three or more vertices is also a cograph.



Considering a complete bipartite graph in which U contains one vertex, and V contains two vertices or less, it is easy to show how such a complete bipartite graph is a cograph. If U contains one vertex and V contains two vertices, the longest induced path is P_3, which is clearly a cograph. Likewise, if both U and V contain one vertex, the longest induced path is P_2, which is also clearly a cograph. The last case is that of the empty set, which is clearly a cograph as well. Thus, all complete bipartite graphs are cographs.