Flow-continuous mappings – influence of the group*

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Abstract

Many questions at the core of graph theory can be formulated as questions about certain group-valued flows: examples are the cycle double cover conjecture, Berge-Fulkerson conjecture, and Tutte's 3-flow, 4-flow, and 5-flow conjectures. As an approach to these problems Jaeger and DeVos, Nešetřil, and Raspaud define a notion of graph morphisms continuous with respect to group-valued flows. We discuss the influence of the group on these maps. In particular, we prove that the number of flow-continuous mappings between two graphs does not depend on the group, but only on the largest order of an element of the group (i.e., on the exponent of the group). Further, there is a nice algebraic structure describing for which groups a mapping is flow-continuous.

On the combinatorial side, we show that for cubic graphs the only relevant groups are $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}$.

1 Introduction

Throughout this paper G and H will be digraphs (finite multidigraphs with loops and parallel edges allowed), $f : E(G) \to E(H)$ a mapping, and M, N abelian groups.

Recall that a mapping $\varphi : E(G) \to M$ is a *flow* (M-flow when we want to emphasize M) when it satisfies the Kirchhoff's law at every vertex, that is, for every $v \in V(G)$ we have

$$\sum_{e \in E(G): e \text{ leaves } v} \varphi(e) = \sum_{e \in E(G): e \text{ enters } v} \varphi(e) \,.$$

The theory of flows on (di)graphs is a very rich one, but also full of longstanding conjectures (cycle double cover, Berge-Fulkerson conjecture, Tutte's 3-flow, 4-flow, and 5-flow conjectures, etc.), see [7], [2], or [9] for a detailed treatment of this area.

In this paper we are going to study a notion introduced by Jaeger [3] and by DeVos, Nešetřil, and Raspaud [1] as an approach to these problems.

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We say that a mapping $f : E(G) \to E(H)$ is M-flow continuous, if "the preimage of every M-flow is an M-flow". More precisely, for every M-flow φ on H, the composition φf (applying first f then φ) is an M-flow on G. For short, we will call M-flow-continuous mappings just FF_M; in the important case $M = \mathbb{Z}_n$ we use the typographically nicer FF_n instead of $FF_{\mathbb{Z}_n}$. We will write $G \xrightarrow{FF_M} H$ to denote that there exists some FF_M mapping from G to H.

The main reason for introducing this notion is Jaeger's conjecture [3] that every bridgeless cubic graph G has a $\mathbb{Z}_2$-flow-continuous mapping to the Petersen graph. If true, this conjecture would imply the cycle double cover conjecture, and many others. In this paper we will study the notion of M-flow-continuous mappings per se, with the aim of making clear what the role of the group M is; this question has not been addressed in previous treatments. For $M = \mathbb{Z}_2$ we do not need to consider the orientation of edges, thus this part of the theory is relevant for undirected graphs. As our emphasis is on general abelian groups, we will mostly deal with digraphs.

In some of our proofs we will use an alternative characterisation of FF-mappings, to state it we need to briefly introduce two notions. Given $\tau : E(G) \to M$ and $f : E(G) \to E(H)$, we denote by τ_f the *algebraic image of* τ, i.e., the mapping $\tau_f : E(H) \to M$ defined by

$$\tau_f(e) = \sum_{e' \in E(G); f(e')=e} \tau(e') \,.$$

A mapping $t : E(G) \to M$ is called an M-*tension* if for every circuit C the sum of t over all clockwise edges is the same as the sum over all counterclockwise edges. It is not hard to see that M-tensions in a plane digraph G correspond to M-flows in the dual G^*. More relevant for us is that for every digraph the vector spaces of all M-flows and of all M-tensions are orthogonal complements. (For this we need M to be a ring. As we will be restricted on finitely generated abelian groups, i.e., on groups in the form (1), this will not limit our use of the following lemma.) This allowed DeVos, Nešetřil, and Raspaud [1, Theorem 3.1] to prove the following useful result.

Lemma 1.1 *Let $f : E(G) \to E(H)$ be a mapping, let M be a* ring. *Mapping f is FF_M if and only if for every M-tension τ on G, its algebraic image τ_f is an M-tension on H.*

Moreover, it is sufficient to verify the condition for all tensions that are nonzero only on a neighborhood of a single vertex.

As an easy corollary of this lemma, we observe that FF_2-mappings between cubic bridgeless graphs map a 3-edge-cut to a 3-edge-cut. In particular, if the target graph is cyclically 4-edge-connected, then the image of an elementary cut (all edges around a vertex) is an elementary cut.

2 Influence of the group

In this section we study how the notion of M-flow-continuous mapping depends on the group M. Although the existence of M-flow-continuous mappings seems to be strongly dependent on the choice of M we prove here (in Theorem 2.4) that this dependence relates only to the largest order of an element of M.

As we are interested only in finite digraphs, we can restrict our attention to finitely generated groups—clearly f is M-flow-continuous if and only if it is N-flow-continuous for every finitely generated subgroup N of M. Hence, there are integers α, k, β_i, n_i ($i = 1, \dots, k$) so that

$$M \simeq \mathbb{Z}^\alpha \times \prod_{i=1}^{k} \mathbb{Z}_{n_i}^{\beta_i} . \tag{1}$$

Note that each such group has a canonical ring structure, thus we will be able to use Lemma 1.1.

For a group M in the form (1), let $n(M) = \infty$ if $\alpha > 0$, otherwise let $n(M)$ be the least common multiple of $\{n_1, \dots, n_k\}$. When $n(M)$ is finite, it is called the *exponent* of the group M. An alternative definition is that $n(M)$ is the largest order of an element of M (here order of $a \in M$ is the smallest $n > 0$ such that $n \cdot a = a + a + \cdots + a = 0$).

As a first step to a complete characterization we consider a specialized question: given a FF_M mapping, when can we conclude that it is FF_N as well?

Lemma 2.1 *1. If f is $FF_{\mathbb{Z}}$ then it is FF_M for any abelian group M.*

2. Let M be a subgroup of abelian group N. If f is FF_N then it is FF_M.

Proof: 1. This appears as Theorem 4.4 in [1].

2. Let φ be an M-flow on H. As $M \le N$, we may regard φ as an N-flow, hence φf is an N-flow on G. As it attains only values in the range of φ, hence in M, it is an M-flow, too. □

Lemma 2.2 *Let M_1, M_2 be two abelian groups. Mapping f is FF_{M_1} and FF_{M_2} if and only if it is $FF_{M_1 \times M_2}$.*

Proof: As M_1, M_2 are isomorphic to subgroups of $M_1 \times M_2$, one implication follows from the second part of Lemma 2.1. For the other implication let φ be an $(M_1 \times M_2)$-flow on H. Write $\varphi = (\varphi_1, \varphi_2)$, where φ_i is an M_i-flow on H. By assumption, $\varphi_i f$ is an M_i flow on G, consequently $\varphi f = (\varphi_1 f, \varphi_2 f)$ is a flow too. □

The following (somewhat surprising) lemma shows that we can restrict our attention to cyclic groups only.

Lemma 2.3 *1. If $n(M) = \infty$ then f is FF_M if and only if it is $FF_{\mathbb{Z}}$.*

2. Otherwise f is FF_M if and only if it is $FF_{n(M)}$.

Proof: In the first part, each implication follows from one part of Lemma 2.1. In the second part: If f is FF_M, we use the fact that $\mathbb{Z}_{n(M)}$ is isomorphic to a subgroup of M, thus the second part of Lemma 2.1 implies f is $FF_{n(M)}$. For the other implication, suppose that f is $FF_{n(M)}$. Note that whenever $\mathbb{Z}_{n_i}$ occurs in the expression (1) for M, then $\mathbb{Z}_{n_i}$ is a subgroup of $\mathbb{Z}_{n(M)}$. Consequently (Lemma 2.1, second part) f is FF_{n_i}. Repeated application of Lemma 2.2 implies f is FF_M as well. □

By a theorem of Tutte [8], for a finite abelian group M, the number of nowhere-zero M-flows on a given (di)graph only depends on the order of M (see also [2, Chapter 6]). Before proceeding in the main direction of this section, let us note a consequence of Lemma 2.3, which is an analogue of Tutte's result.

Theorem 2.4 *Given digraphs G, H, the number of FF_M mappings from G to H depends only on $n(M)$.*

Lemma 2.3 suggests to define for two digraphs the set

$$FF(G,H) = \{n \geq 1 \mid \text{there is } f : E(G) \to E(H) \text{ such that } f \text{ is } FF_n\}$$

and for a particular $f : E(G) \to E(H)$

$$FF(f,G,H) = \{n \geq 1 \mid f \text{ is } FF_n\}\,.$$

We remark that most of these sets contain 1: $\mathbb{Z}_1$ is a trivial group, hence any mapping is FF_1. Therefore $1 \in FF(f,G,H)$ for every $f : E(G) \to E(H)$, while $1 \in FF(G,H)$ if and only if there exists a mapping $E(G) \to E(H)$. This happens always, unless $E(H)$ is empty and $E(G)$ nonempty.

Although we are working with finite digraphs throughout the paper, in the following results we stress this—contrary to most of the other results, these are not true for infinite digraphs.

Lemma 2.5 *Let G be a finite digraph. Either $FF(f,G,H)$ is finite or $FF(f,G,H) = \mathbb{N}$. In the latter case f is $FF_{\mathbb{Z}}$.*

Proof: It is enough to prove that f is $FF_{\mathbb{Z}}$ if it is FF_n for infinitely many integers n. To this end, take a $\mathbb{Z}$-flow φ on H. As $\varphi_n : e \mapsto \varphi(e) \bmod n$ is a $\mathbb{Z}_n$-flow, $\varphi_n f = \varphi f \bmod n$ is a $\mathbb{Z}_n$-flow whenever f is FF_n. To show φf is a $\mathbb{Z}$-flow consider a vertex v of G and let s be the "$\pm$-sum" (in $\mathbb{Z}$) around v:

$$s = \sum_{e \text{ leaves } v} (\varphi f)(e) - \sum_{e \text{ enters } v} (\varphi f)(e)\,.$$

As $s \bmod n = 0$ for infinitely many values of n, we have $s = 0$. □

Any f induced by a local isomorphism is $FF_{\mathbb{Z}}$, thus providing an example where $FF(f,G,H)$ is the whole of $\mathbb{N}$. For finite sets the situation is more interesting. By the next theorem, the sets $FF(f,G,H)$ are precisely the ideals in the divisibility lattice.

Theorem 2.6 *Let S be a finite subset of $\mathbb{N}$. Then the following are equivalent.*

1. *There are G, H, f such that $S = FF(f,G,H)$.*

2. *There is $n \in \mathbb{N}$ such that S is the set of all divisors of n.*

Proof: First we show that 1 implies 2. The set $S = FF(f, G, H)$ has the following properties

(i) If $a \in S$ and $b|a$ then $b \in S$. (We use the second part of Lemma 2.1: if b divides a, then $\mathbb{Z}_b \le \mathbb{Z}_a$.)

(ii) If $a, b \in S$ then the least common multiple of a, b is in S. (We use Lemma 2.1 and Lemma 2.2: if $l = \mathrm{lcm}(a, b)$ then $\mathbb{Z}_l \le \mathbb{Z}_a \times \mathbb{Z}_b$.)

Let n be the maximum of S. By (i), all divisors of n are in S. If there is a $k \in S$ that does not divide n then $\mathrm{lcm}(k, n)$ is an element of S larger than n, a contradiction.

For the other implication, let $\overrightarrow{D}_n$ be a graph with two vertices and n parallel edges in the same direction, let L be a loop (digraph with a single vertex and one edge). Let f be the only mapping from $\overrightarrow{D}_n$ to L. Then $FF(f, \overrightarrow{D}_n, L) = S$: mapping f is FF_k if and only if for any $a \in \mathbb{Z}_k$ the constant mapping $E(\overrightarrow{D}_n) \mapsto a$ is a $\mathbb{Z}_k$-flow; this occurs precisely when k divides n. □

Let us turn to describing the sets $FF(G, H)$.

Lemma 2.7 *Let G, H be* finite *digraphs. Either $FF(G, H)$ is finite or $FF(G, H) = \mathbb{N}$. In the latter case $G \xrightarrow{FF_{\mathbb{Z}}} H$.*

Proof: As in the proof of Lemma 2.5, the only difficult step is to show that if $G \xrightarrow{FF_n} H$ for infinitely many values of n, then $G \xrightarrow{FF_{\mathbb{Z}}} H$. As G and H are finite, there is only a finite number of possible mappings between their edge sets. Hence, there is one of them, say $f : E(G) \to E(H)$, that is FF_n for infinitely many values of n. By Lemma 2.5 we have $f : G \xrightarrow{FF_{\mathbb{Z}}} H$. □

When characterizing the sets $FF(G, H)$ we first remark that an analogue of Lemma 2.2 does not hold: there is a FF_M mapping from $\overrightarrow{D}_9$ to $\overrightarrow{D}_7$ for $M = \mathbb{Z}_2$ (mapping that identifies three edges and is 1–1 on the other ones) and for $M = \mathbb{Z}_3$ (e.g., a constant mapping), but not the same mapping for both, hence there is no $FF_{\mathbb{Z}_2 \times \mathbb{Z}_3}$ mapping. We will see that the sets $FF(G, H)$ are precisely the down-sets in the divisibility poset. First, we prove a lemma that will help us to construct pairs of digraphs G, H with a given $FF(G, H)$. The integer cone of a set $\{s_1, \dots, s_t\} \subseteq \mathbb{N}$ is the set $\{\sum_{i=1}^{t} a_i s_i \mid a_i \in \mathbb{Z}, a_i \ge 0\}$.

Lemma 2.8 *Let A, B be non-empty subsets of $\mathbb{N}$, $n \in \mathbb{N}$, define $G = \bigcup_{a \in A} \overrightarrow{D}_a$, and $H = \bigcup_{b \in B} \overrightarrow{D}_b$. Then there is a FF_n mapping from G to H if and only if*

A is a subset of the integer cone of $B \cup \{n\}$.

Proof: We use Lemma 1.1. Consider a tension τ_a taking the value 1 on $\overrightarrow{D}_a$ and 0 elsewhere. The algebraic image of this tension is a tension, hence it is (modulo n) a sum of several tensions on the digons $\overrightarrow{D}_b$, implying a is in integer cone of $B \cup \{n\}$. On the other hand if $a = \sum_i b_i + cn$ (with $b_i \in B$) then we can map any cn edges of $\overrightarrow{D}_a$

to one (arbitrary) edge of H, and for each i any ("unused") b_i edges bijectively to $\overrightarrow{D}_{b_i}$. After we have done this for each $a \in A$ we will have constructed a $\mathbb{Z}_n$-flow-continuous mapping from G to H. □

Theorem 2.9 *Let S be a finite subset of $\mathbb{N}$. Then the following are equivalent.*

1. *There are G, H such that $S = FF(G,H)$.*
2. *There is a finite set $T \subset \mathbb{N}$ such that*

$$S = \{s \in \mathbb{N}; (\exists t \in T) \quad s|t\}\,.$$

Proof: If S is empty, we take T empty in part 2. In part 1, we just consider digraphs such that $E(H)$ is empty and $E(G)$ is not. Next, we suppose S is nonempty.

By the same reasoning as in the proof of Theorem 2.6 we see that if $a \in FF(G,H)$ and $b|a$ then $b \in FF(G,H)$. Hence, 1 implies 2, as we can take $T = S$ (or, to make T smaller, let T consist of the maximal elements of S in the divisibility relation).

For the other implication let $p > 4 \max T$ be a prime, let $p' \in (1.25p, 1.5p)$ be an integer. Let $A = \{p, p'\}$ and

$$B = \{p - t; t \in T\} \cup \{p' - t; t \in T\}\,;$$

note that every element of B is larger than $\frac{3}{4}p$. As in Lemma 2.8 we define $G = \bigcup_{a\in A} \overrightarrow{D}_a$, $H = \bigcup_{b\in B} \overrightarrow{D}_b$. We claim that $FF(G,H) = S$. By Lemma 2.8 it is immediate that $FF(G,H) \supseteq S$. For the other direction take $n \in FF(G,H)$. By Lemma 2.8 again, we can express p and p' in the form

$$\sum_{i=1}^{k} b_i + cn \tag{2}$$

for integers $c, k \geq 0$, and $b_i \in B$.

- If $k \geq 2$ then the sum in (2) is at least $1.5p$; hence neither p nor p' can be expressed with $k \geq 2$.
- If $k = 1$ then we distinguish two cases.
  - $p = (p - t) + cn$, hence n divides t and thus $n \in S$.
  - $p = (p' - t) + cn$, hence $p' - p \leq t$. But $p' - p > 0.25p > t$, a contradiction.

  Considering p' we find that either $n \in S$ or $p' = (p - t) + cn$.
- Finally, consider $k = 0$. If $p = cn$ then either $n = 1$ (so $n \in S$) or $n = p$. (We don't claim anything about p'.)

To summarize, if $n \in FF(G,H) \setminus S$ then necessarily $n = p$. For p' we have only two possible expressions: $p' = cn$ (for $k = 0$) and $p' = (p - t) + cn$ (for $k = 1$). We easily check that both of them lead to a contradiction. The first one contradicts $1.25p < p' < 1.5p$. In the second expression $c = 0$ implies $p' < p$ while $c \geq 1$ implies $p' \geq 2p - t \geq 1.75p$, again a contradiction. □

Remark 2.10 *This paper concentrates on FF mappings. We remark, however, that analogous proofs describe the role of the group for mappings where preimages of tensions are tensions, or preimages of tensions are flows (or preimages of flows are tensions). For a discussion of the relevance of these types of mappings the reader may consult the series [5, 4] by the authors and the second author's Ph.D. thesis [6].*

3 Cubic graphs

In the previous section we studied how the group M influences the notion of FF_M-mappings; it turned out there is an algebraic structure behind this. In this section we look at the combinatorially more relevant case of cubic graphs. (Degree of each vertex is 3, the orientation is arbitrary.) Indeed, many longstanding conjectures in the area have been reduced to the case of cubic graphs. There it turns out that we only need to consider three groups: $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}$.

Theorem 3.1 *Let $n > 3$ be an integer, suppose G, H are digraphs with maximum degree less than n. Then $G \xrightarrow{FF_n} H$ is equivalent with $G \xrightarrow{FF_{\mathbb{Z}}} H$.*

Proof: One direction follows from Lemma 2.1. For the other one, consider a mapping $f : E(G) \to E(H)$. We will show that if it is FF_n, it is $FF_{\mathbb{Z}}$ as well. Taking a $\mathbb{Z}$-flow φ on H, we will show that φf is a $\mathbb{Z}$-flow on G. We only need to test this on elementary flows (those taking only values ± 1 around a circuit), as these form a basis for $\mathbb{Z}$-flows. So suppose φ is one of these; notice that it is both a $\mathbb{Z}$-flow and a $\mathbb{Z}_n$-flow. Thus, φf is a $\mathbb{Z}_n$-flow on G. Consider a vertex $v \in V(G)$ of degree $d < n$ and let $e_1, e_2, \dots, e_d$ be the edges incident with it; further, let $a_i = \varphi(f(e_i))$. As φf is a $\mathbb{Z}_n$-flow, we have that $s = \pm a_1 \pm a_2 \pm \cdots \pm a_d \equiv 0 \pmod{n}$ (the signs are chosen based on orientation of the edges). Now $|s| \le d < n$, thus $s = 0$. It follows that φf also satisfies the Kirchhoff's law at v in $\mathbb{Z}$, thus φf is also a flow over $\mathbb{Z}$. $\square$

Corollary 3.2 *Let G, H be digraphs of maximum degree 3, let $n > 3$ be an integer. Then $G \xrightarrow{FF_n} H$ is equivalent with $G \xrightarrow{FF_{\mathbb{Z}}} H$.*

Together with Lemma 2.3, the above corollary implies that for subcubic digraphs we only need to consider $\mathbb{Z}_2$-, $\mathbb{Z}_3$-, and $\mathbb{Z}$-flow-continuous mappings.

By Lemma 2.1 $\mathbb{Z}$-flow continuous mapping is also $\mathbb{Z}_2$- and $\mathbb{Z}_3$-flow-continuous. In the following examples we show that existence of FF_2 and FF_3 mappings are independent, even for subcubic digraphs. Let f be any bijection from $E(\overrightarrow{D}_3)$ to $E(\overrightarrow{C}_3)$. Mapping f is FF_n only if n is a multiple of 3, thus it is FF_3 but not FF_2 nor $FF_{\mathbb{Z}}$. On the other hand, consider an edge 3-coloring for $\overrightarrow{K}_4$ (a K_4 with an arbitrary orientation of edges) as a mapping $g : \overrightarrow{K}_4 \to \overrightarrow{D}_3$. This mapping is FF_2 (as a 4-cycle in K_4 is also a cut). However, g is not FF_3: consider a $\mathbb{Z}_3$-flow φ in $\overrightarrow{D}_3$ that equals 1 on all three edges of $\overrightarrow{D}_3$. Clearly the composition φf is not a $\mathbb{Z}_3$-flow on $\overrightarrow{K}_4$.

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