# A NOTE ON CIRCULAR CHROMATIC NUMBER OF GRAPHS WITH LARGE GIRTH AND SIMILAR PROBLEMS

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Abstract. In this short note, we extend the result of Galluccio, Goddyn, and Hell, which states that graphs of large girth excluding a minor are nearly bipartite. We also prove a similar result for the oriented chromatic number, from which follows in particular that graphs of large girth excluding a minor have oriented chromatic number at most 5, and for the $p$th chromatic number $\chi_p$, from which follows in particular that graphs $G$ of large girth excluding a minor have $\chi_p(G) \leq p+2$.

## 1. Introduction

The circular chromatic number $\chi_c(G)$ of a graph $G$, which is a refinement of its chromatic number, has received much attention recently (see [16] for a survey on recent developments).

Recall that the *circular chromatic number* $\chi_c(G)$ of a graph $G$ is the infimum of rational numbers $\frac{n}{k}$ such that there is a mapping from the vertex set of $G$ to $\mathbb{Z}_n$ with the property that for adjacent vertices are mapped to elements at distance $\geq k$.

In general, graphs of large girth can have arbitrary given circular chromatic number:

**Theorem 1** (Nešetřil and Zhu [14]). *For any rational $r \geq 2$ and any positive integers $t, l$, there is a graph $G$ of girth at least $l$ such that $G$ has exactly $t$ $r$-colorings, up to equivalence. In particular, for any $r \geq 2$ and for any integer $l$, there is a graph $G$ of girth at least $l$ which is uniquely $r$-colorable and hence it has $\chi_c(G) = r$.*

However, if restricted to special classes of graphs, large girth graphs may be forced to have small circular chromatic number. For instance:

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**Theorem 2** (Galluccio, Goddyn, and Hell [6])**.** *For any integer $n \geq 4$, for any $\epsilon > 0$, there is an integer $g$ such that every $K_n$-minor free graph $G$ with girth at least $g$ has $\chi_c(G) \leq 2 + \epsilon$.*

The aim of this paper is to extend Theorem 2 to other classes, all with a bounded expansion. Classes with bounded expansion have been introduced in [9, 11], and are based on the requirement for the graph invariant $\nabla_r(G)$ to be bounded in the class for each $r$.

Denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$. Also denote by $|G| = |V(G)|$ (resp. $\|G\| = |E(G)|$) the *order* of $G$ (resp. *size*). Let $G, H$ be graphs with $V(H) = \{v_1, \dots, v_h\}$ and let $r$ be an integer. A graph $H$ is a *shallow minor* of a graph $G$ *at depth* $r$, if there exists disjoint subsets $A_1, \dots, A_h$ of $V(G)$ such that

- the subgraph of $G$ induced by $A_i$ is connected and as radius at most $r$,
- if $v_i$ is adjacent to $v_j$ in $H$, then some vertex in $A_i$ is adjacent in $G$ to some vertex in $A_j$.

We denote [11, 12] by $G \triangledown r$ the class of the (simple) graphs which are shallow minors of $G$ at depth $r$, and we denote by $\nabla_r(G)$ the maximum density of a graph in $G \triangledown r$, that is:

$$\nabla_r(G) = \max_{H \in G \triangledown r} \frac{\|H\|}{|H|}$$

The *expansion* of a class $\mathcal{C}$ is the function $\mathrm{Exp}_{\mathcal{C}} : \mathbb{N} \to \mathbb{R} \cup \{\infty\}$ defined by

$$\mathrm{Exp}_{\mathcal{C}}(r) = \sup_{G \in \mathcal{G}} \nabla_r(G).$$

A class $\mathcal{C}$ has *bounded expansion* if $\mathrm{Exp}_{\mathcal{C}}(r) < \infty$ for each value of $r$.

For instance, the class $\mathcal{D}$ of all graphs with maximum degree 3 has $\mathrm{Exp}_{\mathcal{D}}(r) = (3/2)2^r$, while a class $\mathcal{C}$ has uniformly bounded $\mathrm{Exp}_{\mathcal{C}}$ (that is: $\mathrm{Exp}_{\mathcal{C}}(r) \leq C$ for some constant $C$, independently of $r$) if and only if there is a graph $F$ that is a minor of no graph in $\mathcal{C}$. The expansion function $\mathrm{Exp}_{\mathcal{C}}$ is indeed non-decreasing and no other general constraint exists on the grow rate of expansion functions. In particular, for every non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ with $f(0) = 2$ there exists a class $\mathcal{C}$ with $\mathrm{Exp}_{\mathcal{C}}(r) = f(r)$ (see [12], Exercice 5.1).

However, when studying properties of a class, it is sometimes the case that the frontier between classes that verify the property and those that do not can be (at least approximately) expressed by means of a threshold expansion function (see for instance [5], where it is proved that every class

$\mathcal{C}$ with expansion $\mathrm{Exp}_{\mathcal{C}}(r) \leq c^{r^{1/3-\epsilon}}$ is small, but that some non-small class $\mathcal{C}$ exists, for which $\mathrm{Exp}_{\mathcal{C}}(r) \leq 6 \cdot 3^{\sqrt{r \log(r+e)}}$).

In such a setting, Theorem 2 can be restated as the fact that if $\mathrm{Exp}_{\mathcal{C}}(r) \leq C$ (for some constant $C$) then for any $\epsilon > 0$, there is an integer $g$ such that every $G \in \mathcal{C}$ with girth at least $g$ has $\chi_c(G) \leq 2+\epsilon$. However, such a statement cannot be extended to all classes with bounded expansion, and even to all classes $\mathcal{C}$ with exponential expansion: there exists 3-regular graphs with arbitrary large girth and circular chromatic number at least 7/3 [7], thus the class $\mathcal{D}$ of all graphs with maximum degree 3 (which is such that $\mathrm{Exp}_{\mathcal{D}}(r) = (3/2)2^r$) does not have the property that large girth implies a circular chromatic number arbitrarily close to 2.

We show in this paper (Corollary 1) that exponential expansion is indeed a threshold, in the sense that the condition $\log \mathrm{Exp}_{\mathcal{C}}(r)/r = o(1)$ (as $r \to \infty$) is sufficient to ensure the conclusion of Theorem 2. Hence a natural problem arises, to make this threshold more precise:

*Problem* 1. What is the maximum real integer $c$ such that for every class $\mathcal{C}$ with

$$\lim_{r\to\infty} \frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq c$$

and every positive real $\epsilon > 0$ there is an integer $g$ with the property that every graph $G \in \mathcal{C}$ with girth at least $g$ has $\chi_c(G) \leq 2 + \epsilon$?

It follows from [7] and Corollary 1 that $0 \leq c < \log 2$, and we conjecture $c = 0$.

Other application of our approach are Theorem 5, which concerns the oriented chromatic number and which is the subject of Section 3, and Theorem 7,.which concerns the $p$th chromatic number $\chi_p$ and which is the subject of Section 5.

## 2. Path-degeneracy

One of the main tools used to address homomorphism properties of graphs (or directed graphs) with high-girth is the notion of path-degeneracy (see for instance [6, 13].

**Definition 1.** A graph $G$ is *p-path degenerate* if there is a sequence $G = G_0, G_1, \ldots, G_t$ of subgraphs of $G$ such that $G_t$ is a forest, and each $G_i$ ($i > 0$) is obtained from $G_{i-1}$ by deleting the internal vertices of a path of length at least $p$, all of degree two.

We shall relate path-degeneracy of large girth graphs in a class $\mathcal{C}$ to expansion properties of the class $\mathcal{C}$. First we recall a result that will needed for the proof.

**Theorem 3** (Diestel and Rempel [4])**.** *Let $d$ be an integer and let $G$ be a graph. If* $\mathrm{girth}(G) > 6d+3$ *and* $\delta(G) \geq 3$ *then* $\nabla_{2d}(G) \geq 2^d$

We now explicit the connection between path-degeneracy and expansion properties.

**Lemma 1.** *Let $\mathcal{C}$ be a class of graphs, let $p, g \in \mathbb{N}$, and let $r = \lceil pg/3 \rceil$. If it holds*

$$\frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq \frac{1}{5p}.$$

*Then every graph $G$ in $\mathcal{C}$ with girth at least $g$ is $p$-path degenerate.*

*Proof.* Our proof follows similar lines as the proof ([6], Lemma 2.5). Assume for contradiction that $\mathcal{C}$ include graphs with girth at least $g$ that are not $p$-path degenerate, and let $G$ be a minimal such graph. The graph $G$ is 2-connected, since a graph is $p$-path degenerate if all its blocks are. The graph $G$ is not a circuit since, as $l \geq p+1$, the graph $G$ would be $p$-path degenerate. Hence $G$ is neither an edge or a circuit. Therefore, there exists a unique graph $G'$ with minimum degree at least 3, which is homeomorphic to $G$. The graph $G$ can be obtained from $G'$ by replacing each edge $e \in E(G')$ with a path $P(e)$ of length at least one and at most $(p-1)$. Thus $G' \in \mathcal{C} \triangledown (p-1)/2$. Let $d = (g-3)/6$. According to Theorem 3, if $G'$ has girth greater than $l$ then $\nabla_{2d}(G') \geq 2^d$. As $(\mathcal{C} \triangledown (p-1)/2) \triangledown 2d \subseteq \mathcal{C} \triangledown (2dp + (p-1)/2)$ it holds

$$\nabla_{\lceil lp/3 \rceil}(G) \geq \nabla_{2dp+(p-1)/2}(G) \geq \nabla_{2d}(G') \geq 2^d.$$

Hence it holds

$$\frac{\log \nabla_r(G)}{r} \geq \frac{1}{p} \cdot \frac{d \log 2}{2d+1} > \frac{1}{5p},$$

what contradicts our assumption. □

## 3. Circular Chromatic Number of Graphs with Large Girth

We shall make use of the following property.

**Lemma 2** (Bondy and Hell [1])**.** *For every graph $G$ and integer $k$, the following are equivalent:*

- $\chi_c(G) \leq 2 + \frac{1}{k}$
- $G \to C_{2k+1}$

Existence of a homomorphism to an odd cycle is also linked to path-degenacy.

**Lemma 3** (Galluccio, Goddyn, and Hell [6]). *A $p$-path degenerate graph $G$ admits a homomorphism to any odd circuit of length at most $p+1$.*

*In other words, a $p$-path degenerate graph $G$ has*

$$\chi_c(G) \leq 2 + \frac{1}{\lfloor p/2 \rfloor}.$$

We deduce the following extension of Theorem 2.

**Theorem 4.** *Let $\mathcal{C}$ be a class of graphs and let $\epsilon > 0$. If it holds*

$$\liminf_{r\to\infty} \frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq \frac{\epsilon}{10}.$$

*Then there is an integer $g$ such that every graph $G \in \mathcal{C}$ with girth at least $g$ has $\chi_c(G) \leq 2+\epsilon$.*

*Proof.* Let $p = \lfloor 2/\epsilon \rfloor$. As

$$\liminf_{r\to\infty} \frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} = \liminf_{r\to\infty} \sup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{r} \leq \frac{\epsilon}{10} \leq \frac{1}{5p}$$

there exists $r \in \mathbb{N}$ such that

$$\sup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{r} \leq \frac{1}{5p}.$$

Let $g = \lceil 3r/p \rceil$. According to Lemma 1, every graph $G \in \mathcal{C}$ with girth at least $g$ is $p$-degenerate. Thus, according to Lemma 3, every graph $G \in \mathcal{C}$ with girth at least $g$ is such that $\chi_c(G) \leq 2+\epsilon$. □

**Corollary 1.** *Let $\mathcal{C}$ be a class of graphs such that*

$$\lim_{r\to\infty} \frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} = 0.$$

*Then for every positive real $\epsilon > 0$ there is an integer $g$ such that every graph $G \in \mathcal{C}$ with girth at least $g$ has $\chi_c(G) \leq 2+\epsilon$.*

## 4. Oriented Chromatic Number of Graphs with Large Girth

Recall that the *oriented chromatic number* $\chi_o(G)$ of a (simple) graph $G$ is the minimum $k$ such that every orientation of $G$ admits a homomorphism into some simple digraph with $k$ vertices.

It was proved in [13] that there are planar graphs with arbitrarily large girth having oriented chromatic number 5, and that every planar graph with girth at least 16 has oriented chromatic number at most 5. The bound on the girth was reduced to 13 [3] (where the result generalized to graphs embeddable on the torus, or the Klein bottle) and then to 12 [2]. In [3], it is proved that the considered high-girth graphs not only have oriented chromatic number 5, but that every orientation of these have a homomorphism to the same 5-vertex regular tournament $\vec{C}_5^2$. This tournament is a particular case of the (circular) digraphs $\vec{C}_n^d$, which are defined as the digraphs with vertex set $\mathbb{Z}_n$ and arcs $(i, j)$ when $j - i \in 1, \dots, d \pmod n$.

The following lemma relates $p$-path degeneracy and existence of homomorphism from a directed graph to $\vec{C}_n^d$.

**Lemma 4** (Nešetřil, Raspaud, and Sopena [13])**.** *Then end-vertices of every oriented path $\vec{P}$ of length at least $(n-1)/(d-1)$ can be mapped by a homomorphism $\vec{P} \to \vec{C}_n^d$ to any pair of (not necessarily distinct) vertices of $\vec{C}_n^d$.*

We can now relate expansion properties of a class to the oriented chromatic number of large girth graphs in the class.

**Theorem 5.** *Let $\mathcal{C}$ be a class of graphs, and let $n, d \in \mathbb{N}$. Assume it holds*

$$\frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq \frac{d-1}{5(n-1)}.$$

*Then there is an integer $g$ such that for every orientation $\vec{G}$ of a graph $G$ in $\mathcal{C}$ with girth at least $g$ there exists a homomorphism $\vec{G} \to \vec{C}_n^d$.*

*Proof.* Under the assumptions of the theorem there exists, according to Lemma 1 an integer $g$ such that every graph $G \in \mathcal{C}$ with girth at least $g$ is $(n-1)/(d-1)$-path degenerate. For every orientation $\vec{G}$ of $G$, the existence of a homomorphism $f : \vec{G} \to \vec{C}_n^d$ the easily follows by an iterative use of Lemma 4. □

From this theorem immediately follows:

**Theorem 6.** *Let $\mathcal{C}$ be a class of graphs such that*

$$\limsup_{r\to\infty} \frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq \frac{1}{20}$$

*Then there is an integer $g$ such that every $G \in \mathcal{C}$ with girth at least $g$ has oriented chromatic number at most 5 (and actually are homomorphic to the tournament $\vec{C}_5^2$).*

*Proof.* This follows from Theorem 5, by considering $d = 2$ and $n = 5$. □

In particular, we have

**Corollary 2.** *Let $\mathcal{C}$ be a proper minor closed class of graphs. Then there is an integer $g$ such that every $G \in \mathcal{C}$ with girth at least $g$ has $\chi_o(G) \leq 5$.*

## 5. Centered Coloring of Graphs with Large Girth

Recall that the *tree-depth* $\mathrm{td}(G)$ of a graph $G$ is the minimum height of a rooted forest $F$ such that $G$ is the subgraph of the closure of $F$. For positive integer $p$, the *pth chromatic number* $\chi_p(G)$ is the least number of colors in a vertex coloring of $G$ such that every $i \leq p$ colors induce a subgraph with tree-depth at most $i$. A *p-centered coloring* of a graph $G$ is a vertex coloring such that, for any (induced) connected subgraph $H$ , either some color $c(H)$ appears exactly once in $H$, or $H$ gets at least $p$ colors. For detailed properties of tree-depth and $\chi_p$ we refer the reader to [12]. The $\chi_p$ invariants are related to $p$-centered coloring as follows.

**Lemma 5** (Nešetřil and Ossona de Mendez [10])**.** *For every graph $G$ and every integer $p$, $\chi_p(G)$ is the minimum number of colors in a $(p+1)$-centered coloring of $G$.*

**Lemma 6.** *Let $G$ be a graph and let $p$ be a positive integer. If $G$ is $2p$-path degenerate, then $G$ has a $p$-centered coloring with at most $p+1$ colors.*

*Proof.* According to the definition of $p$-degeneracy, there is a sequence

$$G = G_0 \supset G_1 \supset \cdots \supset G_t$$

of subgraphs of $G$ such that $G_t$ is a forest, and each $G_i$ $(i > 0)$ is obtained from $G_{i-1}$ by deleting the internal vertices of a path $P_i$ of length at least $p$, all of degree two. For each $0 < i \leq t$ select an internal vertex $v_i$ of $P_i$ at distance at least $p$ (in $P_i$) from both endvertices of $P_i$. and orient all the edges of $P_i$ toward $v_i$. Let $C = \{v_i : 0 < i \leq t\}$. Then $G - C$ is a rooted forest, with edges oriented from the roots. Color the vertices $G-C$ with $p$ colors inductively as follows: each root is colored 0, and each other vertex $u$ is colored as $(c+1) \bmod p$, where $c$ is the color assigned to the father of $u$. Note that this coloring is obviously a $p$-centered coloring of $G - C$. Now color the vertices in $C$ with color $p$. It is easily checked that every path linking two vertices in $C$ already got at least $p-1$ colors on the internal vertices by the above coloring of $G-C$. It follows that the every $(p-1)$-colored connected component of $G$ is either included in $G - C$ (hence has a uniquely colored vertex), or contains exactly one vertex in $C$, that is exactly one vertex colored $p$. Thus the defined coloring is a $p$-centered coloring of $G$. □

The following theorem now easily follows.

**Theorem 7.** *Let $\mathcal{C}$ be a class of graphs, and let $p \in \mathbb{N}$. Assume it holds*

$$\frac{\log \mathrm{Exp}_{\mathcal{C}}(r)}{r} \leq \frac{1}{10(p+1)}.$$

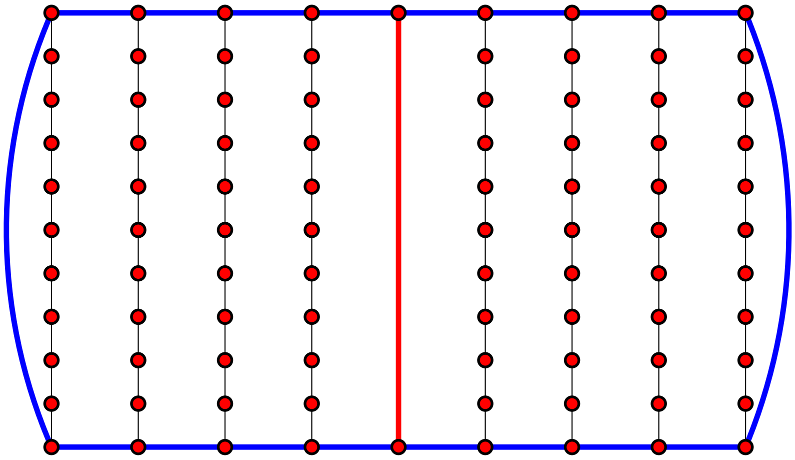
*Then there is an integer $g$ such that every graph $G$ in $\mathcal{C}$ with girth at least $g$ has $\chi_p(G) \leq p + 2$.*

*Proof.* According to Lemma 1 there is an integer $g$ such that every graph $G \in \mathcal{C}$ with girth at least $g$ is $(2p+2)$-path degenerate thus, according to Lemma 6, such that $\chi_p(G) \leq p + 2$. □

In particular, we have

**Corollary 3.** *Let $\mathcal{C}$ be a proper minor closed class of graphs. Then for every integer $p$ there is an integer $g$ such that every $G \in \mathcal{C}$ with girth at least $g$ has $\chi_p(G) \leq p + 2$.*

As $\chi_p(P_n) = p+1$ for sufficiently large $n$, it follows that the upper bound cannot be lowered under $p+1$. The bounds $\chi_1(G) \leq 3$ and $\chi_2(G) \leq 4$ are optimal, as witnessed by a long odd cycle and the series-parallel graphs depicted below, respectively.



We leave as a problem the question whether the bound $p + 2$ is tight for every integer $p$.

## 6. Concluding Remarks

Remark that the property of a class that high girth graphs in the class are nearly bipartite or that they have oriented chromatic number at most 5 is not "topological" in the sense that satisfaction (or non-satisfaction) of this property is not preserved when the graphs in the class are subdivided: Consider an arbitrary class of graphs $\mathcal{C}$. Construct the class $\mathcal{C}'$ as follows: for each graph $G \in \mathcal{C}$ (with $|G|$ vertices) put in $\mathcal{C}'$ the $|G|$-subdivision of $G$. Then the class $\mathcal{C}'$ has the property that high girth graphs in the class are nearly bipartite and have oriented chromatic number at most 5, although this does not have to hold for $\mathcal{C}$. In this sense, considering classes defined by a forbidden minor does not seem to be here optimal.

It is possible, from a class $\mathcal{C}$, to construct the sequence (which we call *resolution* of $\mathcal{C}$) of the classes of shallow minors of graphs in $\mathcal{C}$ at increasing depth

$$\mathcal{C} \subseteq \{G \triangledown 0 : G \in \mathcal{C}\} \subseteq \dots \subseteq \{G \triangledown r : G \in \mathcal{C}\} \subseteq \dots$$

and we note that the results we obtained in this paper are expressed by means of the growth rate of the average degrees of the graphs in these classes, as $r \to \infty$.

A related approach consists in considering the growth rate of the clique number of the graphs in these classes, as $r \to \infty$. For a graph $G$ and an integer $r$ define

$$\omega_r(G) = \max\{\omega(H) : H \in G \triangledown r\}.$$

Then we have the following connection between the grow rate of $\omega_r$ and the existence of sublinear vertex separator in a class of graphs: Then the following holds

**Theorem 8** ([9, 12]). *Let $\mathcal{C}$ be a class of graphs such that*

$$\lim_{r\to\infty} \frac{\log \sup_{G\in\mathcal{C}} \omega_r(G)}{r} = 0.$$

*Then the graphs of order $n$ in $\mathcal{C}$ have vertex separators of size $s(n) = o(n)$ which may be computed in time $O(ns(n)) = o(n^2)$.*

Note that this threshold is sharp, in the sense that for every $c > 0$ there is a class $\mathcal{C}$ with

$$\lim_{r\to\infty} \frac{\log \sup_{G\in\mathcal{C}} \omega_r(G)}{r} \le c$$

but no sublinear vertex separators (consider the class $\mathcal{C}$ of $k$-subdivisions of cubic graphs).

As $\omega_r(G) \le 2\nabla_r(G) + 1$, the classes $\mathcal{C}$ with $\lim_{r\to\infty}(\log \mathrm{Exp}_{\mathcal{C}}(r))/r = 0$, which we considered in this note, satisfy the conditions of Theorem 8. However, the condition on $\omega_r$ is weaker, as witnessed by classes of d-dimensional meshes with bounded aspect ratio (see [8] for a definition), that are such that $\sup_{G\in\mathcal{C}} \omega_r(G)$ grows polynomially with $r$ (see [15]).

We believe that the study of the growth rate of graph parameters on the resolution of a class can be useful to determine threshold where certain class properties stop being true.

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