Bowtie-free graphs have a Ramsey lift

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Abstract

A bowtie is a graph consisting of two triangles with one vertex identified. We show that the class of all (countable) graphs not containing a bowtie as a subgraph have a Ramsey lift (expansion). This is the first non-trivial Ramsey class with a non-trivial algebraic closure.

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1 Introduction

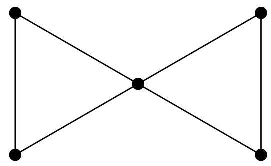

Figure 1: The bowtie graph.

A *bowtie* graph (this poetic name seems to be first used in [23], see also [12], *butterfly graph* or *hourglass graph* are other names used) is formed by two triangles intersecting in a single vertex, see Figure 1. We denote by $\mathcal{B}$ the class of all (countable) graphs not containing a bowtie as a (not necessarily induced) subgraph. The class $\mathcal{B}$ seem to be a rather special and perhaps "unimportant" class. However, it appears that it plays a key role in the context of model theory in the area related to universality and homogeneity. Here is a brief history: The story starts with a (somewhat surprising) result of Komjáth [12] that the class $\mathcal{B}$ contains a *universal graph*, i.e. there exists a (countably infinite) graph U in $\mathcal{B}$ such that any other (whether finite or countably infinite) graph G in $\mathcal{B}$ has an embedding into U. In the other words G is isomorphic to an *induced* subgraph of U. This particular example generated an intensive research (see e.g. [6, 5]). Note that the problem of characterising universal graphs seem to be far from being solved even in the following simple case: Given a finite graph F, denote by $\mathrm{Forb}_M(\mathcal{F})$ the class of all countable graphs not contain F as a (not necessarily induced) subgraph. For which F has the class $\mathrm{Forb}_M(\mathcal{F})$ universal graph? (M stands for monomorphism.) The answer is positive for the bowtie graph while the answer is negative for the rectangle C_4. It is not even known whether this question is decidable, [5].

The problem was recast in the model theory setting by Cherlin et al. [6]. They narrowed the search for universal graphs to more structured ultrahomogeneous and ω-categorical graphs (and structures). In [6] they provided a structural characterisation of ω-categorical universal structures: There is an ω-categorical universal graph in $\mathrm{Forb}_M(\mathcal{F})$ if and only if the class of existentially complete graphs in $\mathrm{Forb}_M(\mathcal{F})$ has a locally finite algebraic closure. And bowtie-free graphs fall in this category.

An important class of this type is the class $\mathrm{Forb}_M(\mathcal{F})$ where $\mathcal{F}$ is a finite set of connected graphs (or relational structures) which is closed on homomorphic images. These classes correspond to the trivial algebraic closure. We equivalently refer to these classes by $\mathrm{Forb}_H(\mathcal{F}')$ where $\mathcal{F}'$ is a set of finite connected structures and $\mathrm{Forb}_H(\mathcal{F}')$ denote the class of all structures not containing a homomorphic image of any $F' \in \mathcal{F}'$.

The bowtie-free graphs have a non-trivial algebraic closure and present in this sense the first non-trivial case (with a non-strong amalgamation). In a sense this obscurely looking example was (and is) a key case for further development (see e.g. [1, 7, 5]).

A different context of universality appeared in the context of Ramsey theory. The Ramsey classes were isolated in 70ies (see [15]) as the top of the line of Ramsey properties and examples found present the backbone of the structural Ramsey theory, see [17, 14, 15]. In [18] the link of Ramsey classes to ultrahomogeneous structures was established: Any Ramsey class is an amalgamation class and thus it is an age of an ultrahomogeneous structure. This was used in [18] to completely characterise hereditary Ramsey classes of undirected graphs. (Essentially, all Ramsey classes were known earlier, [20].) This connection of Ramsey classes proved to be fruitful and led to the characterisation programme for Ramsey classes [16] and to important connection of Ramsey classes with topological dynamics and ergodic theory [11]. This connection is presently treated mostly in the context of model theory. (We review the basic definitions in Section 5.)

In most instances a class is not Ramsey for a trivial reason: One needs to add some information such as ordering or colouring of distinguished parts. For example, all finite graphs form a Ramsey class if we add an ordering of vertices, bipartite graphs need an ordering respecting the bipartition, disjoint union of complete graphs (or equivalences) need an ordering respecting components. This additional information is usually called an expansion, or in a combinatorial setting a lift, of the original structure. Such lifts are usually easy to define, however the example of bowtie-free graphs we present here is an example where the lift is quite complex and uses an intricate system of relations. In fact, it is the first explicitly defined lift; compare [6].

The relationship of Ramsey classes and ultrahomogeneous structures proved to be very inspiring and fruitful. This also motivated search for new Ramsey classes. Using known lists of ultrahomogeneous structures [13, 4, 3] all Ramsey classes of undirected graphs were characterised [18] and, more recently, the class of oriented graphs [10]. Already in Bertinoro 2011 the second author asked whether every ultrahomogeneous structure has an ω-categorical lift which is Ramsey and this led to further research [22, 2]. A typical example is provided by classes $\mathrm{Forb}_H(\mathcal{F})$ where $\mathcal{F}$ is a set of finite connected structures. It has been proved in [6] using model theoretic methods that $\mathrm{Forb}_H(\mathcal{F})$ has an ω-categorical universal object for every finite $\mathcal{F}$ and in [8] those infinite families $\mathcal{F}$ having ω-universal object were characterised. For finite $\mathcal{F}$ the classes $\mathrm{Forb}_H(\mathcal{F})$ were proved to be Ramsey in [19] and, more recently, [9] establishes some further Ramsey classes in this list.

Consequently, this paper is organised as follows: Section 2 contains a detailed description of the structure and the lifts of bowtie-free graphs. This leads to an explicit homogenisation of these graphs which will be used (Section 3). In Section 4 we give a simpler variant of our lift. In Section 5 we review the basic definition of Ramsey classes. The proof of Ramsey property splits into two parts: In Section 6 we prove the Ramsey property for incomplete lifts, and finally in Section 7 we combine this to obtain the final result:

Theorem 1.1 *The class $\mathcal{B}$ has a Ramsey lift (or expansion).*

In a more detailed way this is formulated as Theorem 6.2 below. The final section contains some remarks and open problems.

2 Structure of bowtie-free graphs

In order to prove the Ramsey property one has to understand the expansion very well and the expansion has to be explicit. We start to develop the structure of bowtie-free graphs by means of the following concepts which will describe the structure of triangles in bowtie-free graphs. From now on, in this section, $G = (V, E)$ is a finite bowtie-free graph.

Definition 2.1 (Chimneys) n-chimney graph*, C_n, $n \geq 2$, is a graph created as a free amalgamation of* n *triangles over one common edge. A* chimney graph *is any graph C_n, $n \geq 2$.*

Chimneys together with K_4 (an clique on 4 vertices) will form the only components of bowtie-free graphs formed by triangles (the assumption $n \geq 2$ for chimney is a technical assumption to avoid an isolated triangles).

Definition 2.2 (Good graphs) *A bowtie-free graph $G = (V, E)$ is* good *if every vertex is contained either in a copy of chimney or a copy of the complete graph K_4.*

The structure of bowtie-free graphs is captured by means of the following three lemmas:

Lemma 2.1 *Every bowtie-free graph G is a subgraph of some good graph G'.*

Proof. Graph G can be extended in the following way:

1. For every vertex v not contained in a triangle add new copy of C_2 and identify vertex v with one of vertices of C_2.

2. For every triangle v_1, v_2, v_3 that is not part of a chimney nor K_4 add a new vertex v_4 and triangle v_1, v_2, v_4 turning the original triangle into C_2.

It is easy to see that 1. can not introduce new bowtie.

Assume that 2. introduced a new bowtie. We can assume that v_1 is the unique vertex of degree 4 of this new bowtie and consequently there is another triangle on vertex v_1 in G. Because G is bowtie-free, this triangle must share a common edge with triangle v_1, v_2, v_3 and therefore v_1, v_2, v_3 is already part of K_4 or a chimney in original graph G. A contradiction. □

For a good graph $G = (V, E)$ we split its edge set into two types: $E_0 = E_0(G)$ consisting of all edges in triangles and $E_1 = E_1(G)$ consisting of all remaining edges. We also speak about *edges of type* 0 and *edges of type* 1.

Lemma 2.2 *For every good graph $G = (V, E)$ the graph $G_0 = (V, E_0)$ is a disjoint union of copies of chimneys and K_4.*

Proof. This follows directly from a fact that C_n, $n \geq 2$, and K_4 is a complete listing of bowtie-free graphs with every edge in a triangle. □

Lemma 2.3 *Every good graph G such that graph $G_0 = (V, E_0)$ is as described in Lemma 2.2 and where remaining edges (i.e. edges in E_1) do not close any new triangle is a bowtie-free graph.*

Proof. A bowtie in G must be a bowtie in G_0 and G_0 is a bowtie-free by an assumption of the lemma. □

It follows that bowtie-free graphs are made of chimneys and K_4's (forming the edge set E_0) and a triangle free graph (with the edge set E_1).

An example of a good bowtie-free graph is depicted in Figure 2. Type 0 edges are depicted as solid lines, type 1 edges are dashed.

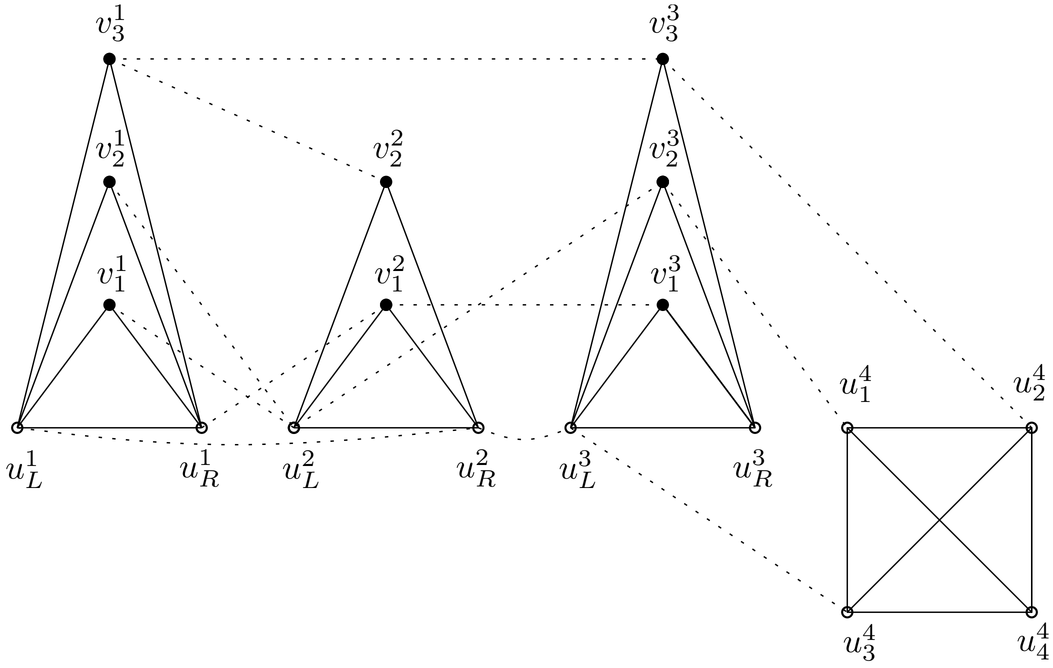

Figure 2: An example of a good graph.

3 (Ultra)Homogenisation of $\mathcal{B}$

A *relational structure* (or simply *structure*) $\mathbf{A}$ is a pair $(A, (R^i_{\mathbf{A}} : i \in I))$, where $R^i_{\mathbf{A}} \subseteq A^{\delta_i}$ (i.e., $R^i_{\mathbf{A}}$ is a δ_i-ary relation on A). We assume that $\delta_i \geq 1$. The *language* L contains names of relations together with their arities. The language is assumed to be understood from the context. However it is the essence of this paper that the languages considered are complex and we consider an interplay of several of them. This will be carefully described. The class of all (countable) relational structures of language L will be denoted by $\mathrm{Rel}(L)$. If the set A is finite we call $\mathbf{A}$ a *finite structure*. We consider only countable or finite structures.

We consider graphs as a special case of relational structure with one binary relation. We use bold letters $\mathbf{A}$, $\mathbf{B}$, ... to denote structures and normal letters G, H, ... for graphs.

A relational structure $\mathbf{A}$ is called *ultrahomogeneous* (or simply *homogeneous*) if every isomorphism between two induced finite substructures of $\mathbf{A}$ can be extended to an automorphism of $\mathbf{A}$. Note that the class of all bowtie-free graphs is defined by the absence of a bowtie as a (non-necessarily induced) subgraph. This is the essentially the only place when we consider monomorphisms. In most places of the paper we consider induced subgraphs and embeddings of structures. Thus we speak about embeddings rather than (induced) subgraphs just to make the distinction more visible.

Let language L' specified by $(\delta'_i; i \in I')$ be a language containing language L. (By this we mean $I \subseteq I'$ and $\delta'_i = \delta_i$ for $i \in I$.) Then every structure $\mathbf{A}' \in \mathrm{Rel}(L')$ may be viewed as a structure $\mathbf{A} = (A, (R^i_{\mathbf{A}}; i \in I)) \in \mathrm{Rel}(L)$ together with some additional relations R^i for $i \in I' \setminus I$.

We call $\mathbf{A}'$ a *lift* (and in the model theory context usually *expansion*; for easy readability we use here both terms) of $\mathbf{A}$. In this situation the structure $\mathbf{A}$ is called the *shadow* (or alternatively the *reduct*) of $\mathbf{A}'$. In this sense the class $\mathrm{Rel}(L')$ is the class of all lifts of $\mathrm{Rel}(L)$. Conversely, $\mathrm{Rel}(L)$ is the class of all shadows of $\mathrm{Rel}(L')$.

A *homogenisation* is a technique which provides a homogeneous lift for a non-homogeneous structure.

The special structure of good graphs indicates that we have vertices of various types and that the Ramsey lift will have to be defined carefully. In this section we shall define three expansions (with languages L_0, L_1 and L_2) and use them to define an amalgamation class (see Corollary 3.1). We start with the definition of central vertices.

Let G be a good graph. Then the *centre* of G, $c(G)$, is a subgraph induced by all vertices contained in two or more triangles.

The *centre of a vertex* v, denoted by $c(v)$, is a subgraph of G induced by all vertices in two or more triangles which are in the same connectivity component of (V, E_0) as the vertex v. (We define centre for good graphs only.) According to Lemma 2.2, the centre of a vertex is either an edge (if v is contained in a chimney) or K_4 (if v is contained in copy of K_4). Note that in the language of model theory this presents a definable set and thus bowtie-free graphs have a nontrivial algebraic closure and this was one of the motivations for a study of this particular example, see e.g. [5]. We call $A \subseteq G$ a *vertex-centre* if and only if $A = c(a)$, for every $a \in A$.

For a good graph G, the *centre* of its subgraph H is a subgraph of G induced by the union of all centres of vertices of H. Note that the centre of H is not necessarily a subgraph of H.

We also call a vertex *central* if it appears in its centre. Other vertices are *non-central.*

Example. The centre of graph depicted in Figure 2 has central vertices labelled u and non-central v. There are 4 vertex centres: $\{u^1_L, u^l_R\}$, $\{u^2_L, u^2_R\}$, $\{u^3_L, u^3_R\}$, and $\{u^4_1, u^4_2, u^4_3, u^4_4\}$. The centre of vertex v^1_1 is $\{u^1_L, u^l_R\}$. The centre of u^4_1 is $\{u^4_1, u^4_2, u^4_3, u^4_4\}$.

We start with the following (easy and optimistic) statement:

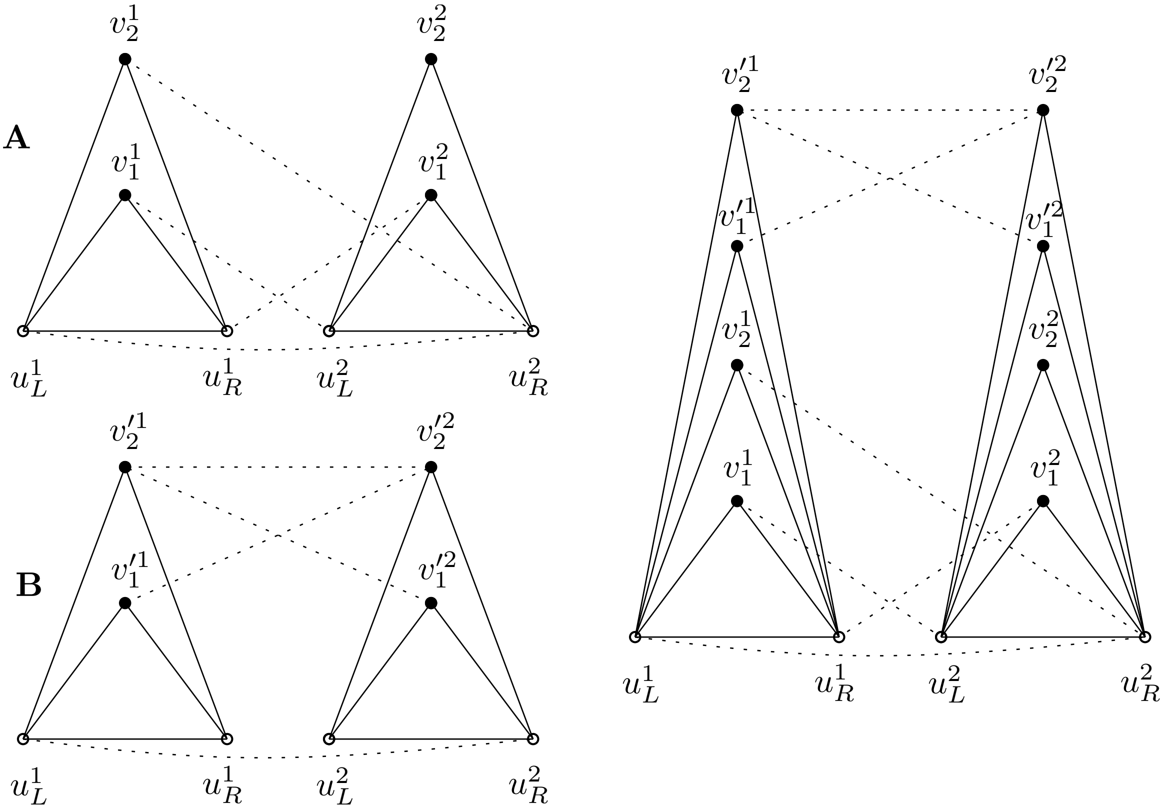

Figure 3: Structures **A** and **B** and their amalgamation over the common centre.

Lemma 3.1 (Central amalgamation) *Let G and G' be good graphs and f an isomorphism from $c(G)$ to $c(G')$. Then the free amalgamation of G and G' over their centres (with respect to f) is a good (bowtie-free) graph.*

Proof. Without loss of generality we can assume that f is an identity and vertex sets of $G = (V, E)$ and $G' = (V', E')$ intersect only on vertices of $c(G)$. The free amalgamation is a graph $G'' = (V \cup V', E \cup E')$.

All triangles of G'' are either triangles in G or G' or both.

G'' is good because all copies of K_4 are part of centres and thus identified. For vertices contained in chimneys, chimney C_n of G gets identified with chimney C_m of G' if an only if centres of the chimneys are the same. This produce a chimney C_{n+m} in G''.

G'' is bowtie-free by Lemma 2.3. □

An example of the amalgamation is depicted in Figure 3.

We consider three expansions corresponding to an increasing chain of languages L_0, L_1, L_2. It is usual that Ramsey classes are classes of ordered structures and thus our first expansion is by orderings:

Definition 3.1 (Expansion L_0) *Given a good graph $G = (V, E)$, an* ordered good graph *is a structure $\mathbf{G} = (V, R_{\mathbf{G}}^{E_0}, R_{\mathbf{G}}^{E_1}, \leq_{\mathbf{G}})$ where $R_{\mathbf{G}}^{E_0} = E_0(G)$, $R_{\mathbf{G}}^{E_1} = E_1(G)$ and $\leq_{\mathbf{G}}$ is a linear order of V such that*

1. *every vertex centre forms an interval of $\leq_{\mathbf{G}}$,*
2. *all centres of chimneys are before vertices in copies of K_4, and,*
3. *all central vertices are before non-central vertices.*

Such ordering is called an admissible ordering. *We denote by L_0 the language of ordered good graphs and by $\mathcal{B}_0$ the class of all ordered good graphs. By an abuse of notation, for a good graphs G we also denote $\mathbf{G} = L_0(G)$ the corresponding ordered good graph (i.e. its L_0 expansion).*

Example. One of admissible orderings of graph in Figure 2 is: u_L^1, u_R^1, u_L^2, u_R^2, u_L^3, u_R^3, u_1^4, u_2^4, u_3^4, u_4^4, v_1^1, v_2^1, v_3^1, v_1^2, v_2^2, v_1^3, v_2^3, v_3^3,

We find it convenient to introduce two more expansions of good graphs. The expansion L_1 is introducing unary relations and expansion L_2 binary relations. This final expansion will produce the lifts which will form the Ramsey class.

Definition 3.2 (Expansion L_1) *Let $\mathbf{G}$ be an ordered good graph. $\mathbf{A} = L_1(\mathbf{G})$ is an expansion of $\mathbf{G}$ adding new unary relations $R_{\mathbf{A}}^L$, $R_{\mathbf{A}}^R$, $R_{\mathbf{A}}^1$, $R_{\mathbf{A}}^2$, $R_{\mathbf{A}}^3$ and $R_{\mathbf{A}}^4$ such that:*

1. *for every $u <_{\mathbf{G}} v$ forming the centre of a chimney of $\mathbf{G}$, we put $(u) \in R_{\mathbf{A}}^L$ and $(v) \in R_{\mathbf{A}}^R$;*
2. *for every $a <_{\mathbf{G}} b <_{\mathbf{G}} c <_{\mathbf{G}} d$ that are vertices of a copy of K_4 in $\mathbf{G}$ we put $(a) \in R_{\mathbf{A}}^1$, $(b) \in R_{\mathbf{A}}^2$, $(c) \in R_{\mathbf{A}}^3$, $(d) \in R_{\mathbf{A}}^4$.*

We denote by L_1 the language of this expansion. For a given $\mathbf{G} \in \mathcal{B}_0$ we denote $L_1(\mathbf{G})$ the corresponding lift of $\mathbf{G}$. By $\mathcal{B}_1$ we then denote the the class of all structures $L_1(\mathbf{G})$, $\mathbf{G} \in \mathcal{B}_0$.

Example. The unary relations of the L_1-expansions of the graph in Figure 2 are indicated by labels of the u vertices.

To advance the definition of the expansion L_2 we first note that we shall sometimes consider *rooted* structures (with either one or two *roots*).

Isomorphisms (and embeddings) are, of course, defined as root preserving isomorphisms (and embeddings). If, for example, the structures $\mathbf{G}$ and $\mathbf{G}'$ are considered with roots u and v and u' and v' then these structures are called isomorphic if there is an isomorphism f from $\mathbf{G}$ to $\mathbf{G}'$ such that $f(u) = u'$ and $f(v) = v'$.

Given ordered good graph $\mathbf{G}$ and two vertices u, v, $u \neq v$, we denote by $t(u, v)$ the isomorphism type of the structure induced by $L_0(\mathbf{G})$ on the set $\{u, v\} \cup c(u) \cup c(v)$ rooted in (u, v). We fix an enumeration $t_1, t_2, \ldots, t_N$ of all the possible such types. Clearly $N \leq 2^{5^2}5^2$ since the size of a centre of a vertex consists of at most 4 vertices. In this situation we define binary relations $R^{t_1}, R^{t_2}, \ldots, R^{t_N}$ as follows:

Definition 3.3 (Homogenising lift L_2) *Let $\mathbf{G} \in \mathcal{B}_0$ be an ordered good graph. $\mathbf{A} = L_2(\mathbf{G})$ is an expansion of L_1-structure $L_1(\mathbf{G})$ adding new binary relations $R^{t_1}_{\mathbf{A}}, R^{t_2}_{\mathbf{A}}, \ldots, R^{t_N}_{\mathbf{A}}$. For $u <_{\mathbf{G}} v$ we put $(u, v) \in R^{t(u,v)}_{\mathbf{A}}$. We denote by L_2 the language of this expansion and by $\mathcal{B}_2$ the the class of all structures $L_2(\mathbf{G})$, $\mathbf{G} \in \mathcal{B}_0$. $L_2(\mathbf{G})$ is called* the lift *of $\mathbf{G}$. (This is our final structure.)*

If $(u, v) \in R^{t(u,v)}_{\mathbf{A}}$ then $t(u, v)$ is called the type *of pair (u, v).*

Let us remark the lift L_2 is a natural homogenisation of good ordered graphs as the new binary relations introduced describe necessary orbits of the automorphism group of a universal graph for class $\mathcal{B}_1$. On the other hand the lift L_1 (i.e. unary relations) was not necessary from the point of view of ω-categoricity. The universal graph constructed in [6] has an automorphisms exchanging vertices within vertex centres. We however consider ordered graphs and the order on every vertex centre prevents any non-trivial automorphism within it. It is interesting to observe that the natural homogenisation of the existentially complete universal graph considered in [6] use only two unary relations (to identify centres of chimneys and vertices of K_4), but it needs relations of unbounded arity. Our ordered lift has only unary and binary relations.

Example. Some types of pairs in the graph depicted in Figure 2 are depicted in Figure 4.

Definition 3.4 *Denote by $\overline{\mathcal{B}}$ the class of all substructures of $\mathcal{B}_2$. For structure $\mathbf{A} \in \overline{\mathcal{B}}$ an ordered good graph $\mathbf{G} \in \mathcal{B}_0$ is called a* witness *of $\mathbf{A}$ if $\mathbf{A}$ is induced on A by $L_2(\mathbf{G})$.*

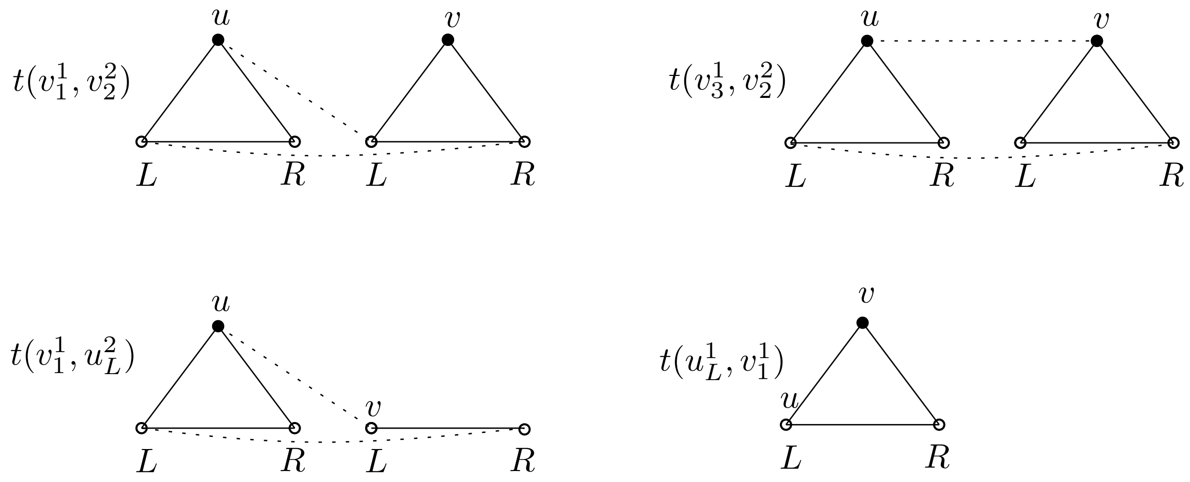

Figure 4: Examples of types of pairs appearing in the graph depicted in Figure 2.

It follows directly for the definition that:

Lemma 3.2 $\mathbf{A} \in \overline{\mathcal{B}}$ *if and only if there exists a witness* $\mathbf{G}$ *of* $\mathbf{A}$*.*

The expansion $L_2(\mathbf{G})$ encode enough information so that for every induced substructure $\mathbf{A}$ of $L_2(\mathbf{G})$ it is possible to uniquely reconstruct the type of its centre (a precise procedure for this appears in proof of Theorem 3.4). As indicated above by Lemma 3.1 this yield the amalgamation property.

Lemma 3.3 (Amalgamation of lifts) $\overline{\mathcal{B}}$ *is an amalgamation class. The amalgamation of* $\overline{\mathcal{B}}$ *is strong on non-central vertices.*

Proof. Fix $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ from $\overline{\mathcal{B}}$ such identity is an embedding from $\mathbf{A}$ to $\mathbf{B}_1$ and $\mathbf{B}_2$. We will construct an amalgam of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

Let $\mathbf{G}_{\mathbf{B}_1}$ and $\mathbf{G}_{\mathbf{B}_2}$ be witnesses of $\mathbf{B}_1$ and $\mathbf{B}_2$ respectively. Denote by $\mathbf{G}_{\mathbf{A}}$ the L_0 reduct of $\mathbf{A}$ (that is an ordered bowtie-free graph). By the construction of the lift, the centre of $\mathbf{G}_{\mathbf{A}}$ in $\mathbf{G}_{\mathbf{B}_1}$ is isomorphic to the centre of $\mathbf{G}_{\mathbf{A}}$ in $\mathbf{G}_{\mathbf{B}_2}$ and moreover there is an isomorphism that is an identity on the central vertices of $\mathbf{A}$. It is now possible to extend $\mathbf{G}_{\mathbf{B}_1}$ to $\mathbf{G}'_{\mathbf{B}_1}$ and $\mathbf{G}_{\mathbf{B}_2}$ to $\mathbf{G}'_{\mathbf{B}_2}$ in a way that $c(\mathbf{G}'_{\mathbf{B}_1})$ and $c(\mathbf{G}'_{\mathbf{B}_2})$ are isomorphic with fixing vertices of $\mathbf{A}$.

By Lemma 3.1 we get ordered good bowtie-free graph $\mathbf{G}_{\mathbf{D}}$ that is an amalgamation of $\mathbf{G}'_{\mathbf{B}_1}$ and $\mathbf{G}'_{\mathbf{B}_2}$ over $\mathbf{G}_{\mathbf{A}}$. It is easy to verify that $L_2(\mathbf{G}_{\mathbf{D}})$ is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$, since the type of every pair of vertices in $\mathbf{B}_1$ or $\mathbf{B}_2$ is preserved and thus $\mathbf{B}_1$ and $\mathbf{B}_2$ are induced substructures of $L_2(\mathbf{G}_{\mathbf{D}})$.

□

Consequently by the standard Fraïssé argument we get:

Corollary 3.1 *The class of all finite structures in $\overline{\mathcal{B}}$ is the age of an ultrahomogeneous structure belonging to $\overline{\mathcal{B}}$ and its shadow is a universal graph for class $\mathcal{B}$.*

This, of course, follows also from [6] where the existence of ω-categorical universal object is established. However, here we provided an explicit expansion by means of the lift structures (which form a finite expansion of bowtie-free graphs).

So we are on a good track nevertheless we need more properties, gaining more information about $\overline{\mathcal{B}}$. Particularly, we need the following alternative description of $\overline{\mathcal{B}}$. This lemma will allow us to use strong Ramsey properties proved in [20].

Theorem 3.4 *Class $\overline{\mathcal{B}}$ can equivalently be described as the class of all those complete structures in* $\mathrm{Rel}(L_2)$ *not containing a set $\mathcal{T}$ of complete L_2-structures on at most* 3 *vertices as substructures. I.e. there is a finite set $\mathcal{T}$ of complete structures on at most 3 vertices such that $\overline{\mathcal{B}}$ is the class of all complete structures belonging to* $\mathrm{Forb}(\mathcal{T})$.

Recall that a structure $\mathbf{A}$ is called *complete* (in [21] *irreducible*) if every pair of distinct vertices belong to a relation of $\mathbf{A}$. By $\mathrm{Forb}(\mathcal{F})$ we denote class of structures not containing any structure $\mathbf{F} \in \mathcal{F}$ as an induced substructure.

Proof. Consider structure $\mathbf{A} \in \overline{\mathcal{B}}$. It easily follows from Definition 3.4 that every pair of vertices (u, v), $u \leq_{\mathbf{A}} v$, is in some binary relation $R^{t_j}_{\mathbf{A}}$ and thus $\overline{\mathcal{B}}$ is a class of complete structures.

Because $\overline{\mathcal{B}}$ is closed on induced substructures it remains to show that every $\mathbf{A} \notin \overline{\mathcal{B}}$ contains substructure $\mathbf{A}' \notin \overline{\mathcal{B}}$ that consist of at most 3 vertices (see Figure 4).

We give an effective procedure that attempts to construct, for a given structure $\mathbf{A}$, a good ordered bowtie graph (witness) $\mathbf{G}$ such that $\mathbf{A}$ is an induced substructure of $L_2(\mathbf{G})$. By Lemma 3.2 the existence a such graph $\mathbf{G}$ prove that $\mathbf{A} \in \overline{\mathcal{B}}$. We analyse cases where such procedure fails and show that these failures all correspond structures $\mathbf{F}$ on at most 3 vertices. All those structures have property that $\mathbf{F} \notin \overline{\mathcal{B}}$.

Denote by $\mathbf{A}^0$ the L_1-reduct of $\mathbf{A}$. Enumerate all pairs of vertices u, v, $u <_{\mathbf{A}} v$ in A as $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$. For every pair (u_i, v_i), $1 \leq i \leq n$, we construct $\mathbf{A}^i$ inductively from $\mathbf{A}^{i-1}$ based on the type of (u_i, v_i) in $\mathbf{A}$. This involves the following elementary steps:

1. addition of new vertices to represent centres of u and v if they are not already present in $\mathbf{A}^{i-1}$,
2. addition of the new vertices into the corresponding unary relations $R^L_{\mathbf{A}^i}$, $R^R_{\mathbf{A}^i}$, $R^1_{\mathbf{A}^i}$, $R^2_{\mathbf{A}^i}$, $R^3_{\mathbf{A}^i}$, and, $R^4_{\mathbf{A}^i}$ as required by the type,
3. addition of new edges of type 0 or 1 from u and v to the newly added vertices,
4. addition of edges of type 0 or 1 between the newly added vertices,
5. extension of the linear order $\leq_{\mathbf{A}^{i-1}}$ in a way consistent with the definition of ordered good graphs (Definition 3.1) and the type of the pair (u, v).

Because vertices of centres of a given vertex v are uniquely determined by the unary relations in L_1, there is (up to isomorphism) unique way of doing so (if it exist at all).

This procedure may fail if the extension is impossible. Assume that (u_i, v_i) is the first pair such that $\mathbf{A}^i$ can not be constructed. We consider individual cases that may happen and show that such failure scenarios all imply existence of a forbidden triangles or digons in $\mathbf{A}$:

1. Some or all vertices of centre of u already exists in $\mathbf{G}^{i-1}$ and they are in conflict with the centre required by the type of pair (u, v).

   For example there is a vertex u' connected by edge of type 0 to u that is in $R^L_{\mathbf{A}}$ while the centre of u required is a copy of K_4 and thus the vertex should be in $R^1_{\mathbf{A}}$, $R^2_{\mathbf{A}}$, $R^3_{\mathbf{A}}$ or $R^4_{\mathbf{A}}$ instead.

   In this case let u' be such vertex. If u' is in A then the structure induced on u, v, u' must be forbidden: pair (u, v) require u to have its centre of one type, while pair (u, u') require its centre of a different type (or if $u = u'$ then unary the relation on u must be already in conflict). This is not possible in structure in $\overline{\mathcal{B}}$.

   If u' is not in A then it was introduced when defining the centre of vertex u'' and then u, v, u'' induce the forbidden substructure for the same reason.

2. The centre of v is already defined and different than one required by the type.

   This case follows in complete analogy to 1.

3. Vertices u and v are connected or ordered differently than required by the type. In this case the structure induced on u, v is forbidden.

4. Edges or orders in between already defined parts of centres u and v are different then required by the type.

   Denote by u' and v' the conflicting vertices of the the centre of u and v respectively. Now put $u'' = u'$ if $u' \in A$ or put u'' to be a vertex of A whose centre contains u'. Similarly put $v'' = v$ if $v' \in A$ or v'' to a vertex of A whose centre contains v'. Now structure induced on u, v, u'', v'' is forbidden and moreover at least one of u, v, u'' or u'', v, v'' must be forbidden.

We have shown that the procedure of adding centres can always be completed for all pairs $(u_i, v_i), i = 1, 2, \ldots, n$ if all substructures of $\mathbf{A}$ on at most 3 vertices are in $\overline{\mathcal{B}}$. Denote by $\mathbf{G}^n$ the resulting ordered graph (i.e. L_0 reduct of $\mathbf{A}^n$). By the construction, $\mathbf{G}^n$ is an ordered good graph. It remains to verify that $\mathbf{G}^n$ contains no bowtie. We use Lemma 2.3. Assume, for the contrary, that $\mathbf{G}^n$ contains a triangle u, v, w such that u, v is an edge in $R^{E_1}_{\mathbf{G}^n}$ and other edges are in $R^{E_0}_{\mathbf{G}^n}$ or $R^{E_1}_{\mathbf{G}^n}$. Again we consider a triple of vertices $u', v', w' \in A$ such that u, v, w are either equivalent to u', v', w' or present in their centres. u', v', w' again induce a forbidden substructure of $\mathbf{A}$.

It follows that we can characterise lifts $\mathbf{A}$ such that there exists $\mathbf{A}^n$ described above that is an ordered good bowtie-free graph. Because $\mathbf{A}$ is substructure of $L_2(\mathbf{A}^n)$ (and thus $\mathbf{A} \in \overline{\mathcal{B}}$) the statement follows. $\square$

4 Reduced structures

To simplify our future analysis, we now invoke another modification of good L_2-structures. It is easy to see that the L_2-lifts was created in a way so all edges in between two vertex centres, C_1 and C_2, are in fact encoded by an L_2-edge in between any pair of vertices $v_1 \in C_1$ and $v_2 \in C_2$. We can thus safely omit all but one vertex from every vertex-centre without losing any information about a good L_2-structure:

Definition 4.1 (Reduced structures) $\mathbf{A}^\bullet$ *is a* reduction *of good L_2-structure* $\mathbf{A}$ *if it is created from* $\mathbf{A}$ *by removing all vertices* $v \in R^R_{\mathbf{A}}, R^2_{\mathbf{A}}, R^3_{\mathbf{A}}, R^4_{\mathbf{A}}$.

These reduced structures are still described by a set of forbidden substructures on at most 3 vertices (in a sense of Theorem 3.4). We modify the language L_2 correspondingly into $L^\bullet$ and denote by $\mathcal{B}^\bullet$ the class of all reduced structures $\mathbf{A}^\bullet$ where $\mathbf{A} \in \mathcal{B}$.

The class $\mathcal{B}^\bullet$, can thus be described as the class of all complete ordered structures belonging to $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ where $\mathcal{T}$ is a finite set of complete structures each with at most 3 vertices.

5 Ramsey structures

To make this paper self-contained, let us introduce (by now standard) notation (see e.g. [15]):

Let $\mathcal{C}$ be a class of structures endowed with embeddings. The class is usually understood from the context. Let $\mathbf{A}, \mathbf{B}$ be objects of $\mathcal{C}$. Then by $\binom{\mathbf{B}}{\mathbf{A}}$ we denote the set of all sub-objects $\widetilde{\mathbf{A}}$ of $\mathbf{B}$, $\widetilde{\mathbf{A}}$ isomorphic to $\mathbf{A}$. (By a sub-object we mean that the inclusion is an embedding.) Using this notation the definition of Ramsey class gets the following form:

A class $\mathcal{C}$ is a *Ramsey class* if for every its two objects $\mathcal{A}$ and $\mathcal{B}$ and for every positive integer k there exists object $\mathcal{C}$ such that the following holds: For every partition $\binom{\mathbf{B}}{\mathbf{A}}$ in k classes there exists $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition.

It is usual to shorten the last part of the definition as $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. The following strong Ramsey theorem is the main result of [20]. It will be used repeatedly (for example in Sections 6 and 7).

Theorem 5.1 ([20]) *Let L be a finite relational language involving unary relations $U_1, U_2, \ldots, U_N$. Let $\mathcal{F}$ be a set of complete L-structures. Let $\mathcal{C}$ be the class of all ordered L-structures $\mathbf{A}$ where the ordering $\leq_{\mathbf{A}}$ satisfies*

$$x < y \text{ whenever } (x) \in R^{U_i}_{\mathbf{A}} \text{ and } (y) \in R^{U_j}_{\mathbf{A}} \text{ and } 1 \leq i < j \leq N.$$

We call such ordering an admissible ordering. *Then the class* $\mathrm{Forb}(\mathcal{F})$ *together with admissible orderings is a Ramsey class.*

In fact, this is an N-partite version of the main result of [20]. It follows easily either from the (partite construction) proof, or directly by a product argument.

We apply this theorem to a special class $\mathcal{T}$ of irreducible structures isolated in the Theorem 3.4. Denote by $\mathcal{T}$ ($\mathcal{T}$ for triangles) the set of all relational structures on at most 3 vertices for which the class of all complete structures in $\mathrm{Forb}_{L_2}(\mathcal{T})$ coincides with the class $\overline{\mathcal{B}}$ (see Theorem 3.4). We consider the structures in $\mathrm{Forb}_{L_2}(\mathcal{T})$ with orderings defined above (which are admissible). It is also more convenient to consider the contracted versions $L^\bullet$ of the language L_2. In this section we prove:

Theorem 5.2 *The class* $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ *is a Ramsey class.*

Explicitly: For every $L^\bullet$*-structures* $\mathbf{A}, \mathbf{B}$ *in* $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ *there exists a structure* $\mathbf{C} \in \mathrm{Forb}_{L^\bullet}(\mathcal{T})$ *such that*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Proof. Indeed this is just a specialisation of the above result Theorem 5.1. □

However note that even when $\mathbf{A}$ and $\mathbf{B}$ are complete structures the structure $\mathbf{C}$ in $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ is not necessarily complete and thus it need not correspond to the lift of a bowtie-free graph. Theorem 5.2 is just the first step in proving the main result.

6 Star Equivalences are Ramsey

The key feature of bowtie-free graphs is the partition to chimneys with each class of partition "rooted" in the centre (the root being its algebraic closure). In this section we prove the Theorem 6.2 which extends the Theorem 5.2 to structures with such equivalences and thus brings us closer to the main result (which is proved in the next section).

Definition 6.1 (Chimney equivalence) *For an* $L^\bullet$*-structure* $\mathbf{A} \in \mathrm{Forb}_{L^\bullet}(\mathcal{T})$ *which is complete (i.e. which is the lift of a good bowtie-graph) denote by* $\sim_{\mathbf{A}}$ *the equivalence expressing that two vertices belong to the same chimney (contracted central vertices are included in this).* $\sim_{\mathbf{A}}$ *is called the* chimney equivalence *of* $\mathbf{A}$.

Note that each equivalence class of $\sim_{\mathbf{A}}$ contains a distinguished vertex x—the (reduced) centre of the corresponding chimney. Moreover all other vertices of this equivalence class are related to x by edges belonging to $R^{E_0}_{\mathbf{A}}$ that corresponds to a (spanning) star and there are no other vertices joined to x by $R^{E_0}_{\mathbf{A}}$ edges. Thus the equivalence $\sim_{\mathbf{A}}$ corresponds with that star system formed by $R^{E_0}_{\mathbf{A}}$ edges. This leads us to the following definition which make sense for incomplete structures in $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$:

Definition 6.2 (Star equivalence) *For an $L^\bullet$-structure $\mathbf{A} \in \mathrm{Forb}_{L^\bullet}(\mathcal{T})$ assume that the edges $L^0_{\mathbf{A}}$ form a star forest. Denote by $\approx_{\mathbf{A}}$ (called* star equivalence*) the equivalence expressing the component structure of this star forest.*

The equivalence $\approx_{\mathbf{A}}$ for incomplete systems will play the role of the chimney equivalence for complete systems.

Definition 6.3 *Denote by $\mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$ the class of all structures $\mathbf{A} \in \mathrm{Forb}_{L^\bullet}(\mathcal{T})$ where $\approx_{\mathbf{A}}$ is an star equivalence and such that all central central vertices in $R^L_{\mathbf{A}}$ appears as centres of the stars (possibly degenerated to 1 vertex). The ordering is inherited from the structures in $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$.*

In this section we aim to prove Theorem 6.2 which gives Ramsey property for structures with star equivalences.

We shall stress the fact that $\mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$ can not be expressed as a class $\mathrm{Forb}_{L^\bullet}(\mathcal{T}')$ where $\mathcal{T}'\backslash\mathcal{T}$ is a set of some irreducible structures. There is no way to express the fact that no vertex can be connected to centres of two different stars. Consequently we can not apply Theorem 5.1 directly. The proof of this is a variant of the Partite Construction [18]. We modify its core part—Partite Lemma—in order to satisfy the additional equivalence condition. This strengthening of the Partite Lemma has some further consequences however in this paper we formulate it just in the context of our main theorem about bowtie-free graphs.

The following is the central definition for this section.

Definition 6.4 (A-partite system) *Let $\mathbf{A}$ be a $L^\bullet$-system. Assume $A = \{1, 2, \ldots, a\}$ with the natural ordering. An $\mathbf{A}$-partite $L^\bullet$-system is a tuple $(\mathbf{A}, \mathcal{X}_{\mathbf{B}}, \mathbf{B})$ where $\mathbf{B}$ is an $L^\bullet$-structure and $\mathcal{X}_{\mathbf{B}} = \{X^1_{\mathbf{B}}, X^2_{\mathbf{B}}, \ldots, X^a_{\mathbf{B}}\}$ partitions vertex set of $\mathbf{B}$ into a classes ($X^i_{\mathbf{B}}$ are called* parts *of $\mathbf{B}$) such that*

1. *the ordering of $\mathbf{B}$ is lexicographic induced by the ordering of $\mathbf{A}$ and of parts $X^i_{\mathbf{B}}$ (particularly it satisfies $X^1_{\mathbf{B}} < X^2_{\mathbf{B}} < \ldots < X^a_{\mathbf{B}}$);*
2. *mapping π which maps every $x \in X^i_{\mathbf{B}}$ to i ($i = 1, 2, \ldots, a$) is a monotone homomorphism in $L^\bullet$ (π is called the* projection*);*
3. *every tuple in every relation of $\mathbf{B}$ meets every class $X^i_{\mathbf{B}}$ in at most one element (i.e. these tuples are transversal with respect to the partition).*

The isomorphisms and embeddings of $\mathbf{A}$*-partite structures, say of* $\mathbf{B}$ *into* $\mathbf{B}'$ *are defined as the isomorphisms and embeddings of structures together with the condition that all parts are being preserved (the part* $X^i_{\mathbf{B}}$ *is mapped to* $X^i_{\mathbf{B}'}$ *for every* $i = 1, 2, \ldots, a$*).*

In the following we will consider $L^\bullet$ structure $\mathbf{A}$ to be also an $\mathbf{A}$-partite system, where each of the partitions consist of single vertex. We start by proving the following modification of the Partite Lemma [18]:

Lemma 6.1 (Partite Lemma) *Let* $\mathbf{A}$ *be a good* $L^\bullet$*-structure with star equivalence* $\approx_{\mathbf{A}}$ *(i.e. we assume that* $\mathbf{A}$ *is a reduction of a good* L_2*-structure). Assume without loss of generality* $A = \{1, 2, \ldots, a\}$ *with the natural ordering. Let* $\mathbf{B}$ *be an* $\mathbf{A}$*-partite* $L^\bullet$*-system with parts* $\mathcal{X}_{\mathbf{B}} = \{X^1_{\mathbf{B}}, X^2_{\mathbf{B}}, \ldots, X^a_{\mathbf{B}}\}$ *and star equivalence* $\approx_{\mathbf{B}}$*. Further assume that every vertex of* $\mathbf{B}$ *is contained in a copy of* $\mathbf{A}$*. Then there exists an* $\mathbf{A}$*-partite* $L^\bullet$*-system* $\mathbf{C}$ *with parts* $\mathcal{X}_{\mathbf{C}} = \{X^1_{\mathbf{C}}, X^2_{\mathbf{C}}, \ldots, X^a_{\mathbf{C}}\}$ *and star* $\approx_{\mathbf{C}}$ *such that*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2.$$

Explicitly: For every 2-colouring of all substructures of $\mathbf{C}$ *which are isomorphic to* $\mathbf{A}$ *there exists a substructure* $\widetilde{\mathbf{B}}$ *of* $\mathbf{C}$*,* $\widetilde{\mathbf{B}}$ *isomorphic to* $\mathbf{B}$*, such that all the substructures of* $\widetilde{\mathbf{B}}$ *which are isomorphic to* $\mathbf{A}$ *are all monochromatic. Particularly, the isomorphism of* $\widetilde{\mathbf{B}}$ *and* $\mathbf{B}$ *(which is an embedding of* $\mathbf{B}$ *into* $\mathbf{C}$*) maps the equivalence* $\approx_{\mathbf{B}}$ *into* $\approx_{\mathbf{C}}$*.*

Proof. Let $\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_t$ be the enumeration of all substructures of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$.

We take N sufficiently large (that will be defined later) and construct an $\mathbf{A}$-partite $L^\bullet$-system $\mathbf{C}$ with parts $\mathcal{X}_{\mathbf{C}} = \{X^1_{\mathbf{C}}, X^2_{\mathbf{C}}, \ldots, X^a_{\mathbf{C}}\}$ as follows:

1. For every $1 \le i \le a$ set $X^i_{\mathbf{C}}$ is the set of all functions

$$f : \{1, 2, \ldots, N\} \to X^i_{\mathbf{B}}.$$

2. $C = \bigcup_{i=1}^a X^i_{\mathbf{C}}$ with ordering satisfying $X^1_{\mathbf{C}} < X^2_{\mathbf{C}} < \ldots < X^a_{\mathbf{C}}$.
3. For every relation j of $L^\bullet$, $(f_1, f_2, \ldots, f_r) \in R^j_{\mathbf{C}}$ if and only if one of the following occur:
   (a) There exists function $u : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, t\}$ such that for every $1 \le l \le N$ the tuple $(f_1(l), f_2(l), \ldots, f_r(l))$ is in $R^j_{\widetilde{\mathbf{A}}_{u(l)}}$.

(b) There exists $\omega \subseteq \{1, 2, \ldots, N\}$ and function $u : \{1, 2, \ldots, N\} \setminus \omega \to \{1, 2, \ldots, t\}$ such that functions f_1, f_2, …, f_r are all constant on ω and:

   i. $f_1(l)$, $f_2(l)$, …, $f_r(l)$ are all vertices of $\widetilde{\mathbf{A}}_{u(l)}$ in $\mathbf{B}$, for every $l \in \{1, 2, \ldots, N\} \setminus \omega$,

   ii. $(f_1(l), f_2(l), \ldots, f_r(l)) \in R^j_{\mathbf{B}}$ and there is no copy of $\mathbf{A}$ in $\mathbf{B}$ containing all vertices $f_1(l), f_2(l), \ldots, f_r(l)$, for $l \in \omega$.

The ordering of $\mathbf{C}$ is defined lexicographically as an extension of all orderings of $\mathbf{A}$ and $\{1, 2, \ldots, N\}$.

If vertex v of $\mathbf{B}$ is contained in the star S, denote by $s_{\mathbf{B}}(v)$ the centre of S and put $s_{\mathbf{B}}(v) = v$ otherwise. Define the equivalence $\sim$ on $\mathbf{C}$ as follows: $f \sim g$ if and only if $s_{\mathbf{B}}(f(l)) = s_{\mathbf{B}}(g(l))$ for every $1 \leq l \leq N$.

We shall check that indeed $\mathbf{C}$ is an $\mathbf{A}$-partite $L^\bullet$-system with parts $\mathcal{X}_{\mathbf{C}} = \{X^1_{\mathbf{C}}, X^2_{\mathbf{C}}, \ldots, X^a_{\mathbf{C}}\}$ and finally we prove that the star equivalence $\approx_{\mathbf{C}}$ coincides with $\sim$. Most of this follows immediately from the definition (and it holds for any relational structure). We pay extra attention to checking that $\sim$ give the star equivalency. This is the main difference from [18].

It is easy to see that $\sim$ is indeed an equivalence. We show that the $\sim$ is the star equivalence of $\approx_{\mathbf{C}}$. First observe that $u \approx_{\mathbf{B}} v$ if and only of $s_{\mathbf{B}}(u) = s_{\mathbf{B}}(v)$. Moreover, because $\mathbf{A}$ corresponds to a reduced good structure and because every vertex of $\mathbf{B}$ is contained in a copy of $\mathbf{A}$ we know that every edge of type 0 in $\mathbf{B}$ is an edge of a copy of $\mathbf{A}$. It follows that vertices f, g of $\mathbf{C}$ are connected by edge of type 0 if and only if for each $1 \leq l \leq N$ we have an edge of type 0 in between $f(l)$ and $g(l)$, and consequently we have $s_{\mathbf{B}}(f(l)) = s_{\mathbf{B}}(g(l))$. Thus $\approx_{\mathbf{C}} = \sim$.

We check that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. Let N be the Hales-Jewett number guaranteeing a monochromatic line in any 2-colouring of N-dimensional cube over alphabet $\{1, 2, \ldots, t\}$.

Now assume that $\mathcal{A}_1, \mathcal{A}_2$ is a 2-colouring of all copies of $\mathbf{A}$ in $\mathbf{C}$. Using the definition of $\mathbf{C}$ we see that among these copies of $\mathbf{A}$ are copies induced by an N-tuple $(\widetilde{\mathbf{A}}_{u(1)}, \widetilde{\mathbf{A}}_{u(2)}, \ldots, \widetilde{\mathbf{A}}_{u(N)})$ of copies of $\mathbf{A}$ for every function $u : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, t\}$. However such copies are coded by the elements of the cube $\{1, 2, \ldots, t\}^N$ and thus there is a monochromatic combinatorial line. This line in turn will lead to a copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}$ with all edges of the form (a), (b) described above. □

We can now invoke the Partite Construction [18] in its standard form. We prove:

Theorem 6.2 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be good* $L^\bullet$*-structures with star equivalences* $\approx_\mathbf{A}$ *and* $\approx_\mathbf{B}$*. Thus we are assuming that* $\mathbf{A}, \mathbf{B} \in \mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$*. Then there exists* $L^\bullet$*-structure* $\mathbf{C} \in \mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{F})$ *with the star equivalence* $\approx_\mathbf{C}$ *such that*

$$\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$$

with respect to embeddings preserving the equivalences.

Proof. Fix structures $\mathbf{A}, \mathbf{B} \in \mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$. Consider $\mathbf{A}$ and $\mathbf{B}$ as structures in $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ and using Theorem 5.2 obtain $\mathbf{C}_0$ that satisfies $\mathbf{C}_0 \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. Assume without loss of generality that $C_0 = \{1, 2, \ldots, c\}$. Enumerate all copies of $\mathbf{A}$ in $\mathbf{C}_0$ as $\{\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_b\}$. We shall define $\mathbf{C}_0$-partite structures $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_b$ which, as we shall show, all belong to $Forb^{\approx}_{L^\bullet}(\mathcal{T})$. Putting $\mathbf{C} = \mathbf{P}_b$ we shall have the desired Ramsey property $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$.

Put explicitly $\mathcal{X}_{\mathbf{P}_i} = \{X^i_1, X^i_2, \ldots, X^i_c\}$. As usual the systems $\mathbf{P}_i$ are called *pictures*. They will be constructed by induction on i.

The picture $\mathbf{P}_0$ is constructed as a disjoint union of copies of $\mathbf{B}$: for every copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}_0$ we consider a new isomorphic and disjoint copy $\widetilde{\mathbf{B}}'$ in $\mathbf{P}_0$ which intersects part X^0_l if and only if $\widetilde{\mathbf{B}}$ (so the projection of $\widetilde{\mathbf{B}}'$ is $\widetilde{\mathbf{B}}$).

Let the picture $\mathbf{P}_i$ be already constructed. Let $\widetilde{A}_i = \{x_1, x_2, \ldots, x_a\}$ be the vertices of $\widetilde{\mathbf{A}}_i$ (in the order of $\mathbf{C}_0$). Let $\mathbf{B}_i$ be the substructure of $\mathbf{P}_i$ induced by $\mathbf{P}_i$ on the union of vertices of those copies of $\mathbf{A}$ which projects to $\widetilde{\mathbf{A}}_i$. (Note that $\mathbf{B}_i$ does not have all vertices of $\mathbf{P}_i$ contained in parts $X^i_{x_1}, X^i_{x_2}, \ldots, X^i_{x_a}$). In this situation we use Partite Lemma 6.1 to obtain an $\mathbf{C}_0$-partite system $\mathbf{C}_{i+1}$ with parts $X^{i+1}_{x_1}, X^{i+1}_{x_2}, \ldots, X^{i+1}_{x_a}$. Now consider all substructures of $\mathbf{C}_{i+1}$ which are isomorphic to $\mathbf{B}_i$ and extend each of these structures to a copy of $\mathbf{P}_i$. These copies are disjoint outside $\mathbf{C}_{i+1}$, however in this extension we preserve the parts of all the copies. The result of this multiple amalgamation is $\mathbf{P}_{i+1}$.

Put $\mathbf{C} = \mathbf{P}_b$. It follows easily that $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$: by a backward induction one proves that in any 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists a copy $\mathbf{P}$ of $\mathbf{P}_0$ such that the colour of a copy of $\mathbf{A}$ in $\mathbf{P}$ depends only on its projection. As this in turn induces colouring of copies of $\mathbf{A}$ in $\mathbf{C}_0$, we obtain a monochromatic copy of $\mathbf{B}$.

We have to check that $\mathbf{C}$ belongs to $\mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$. To do so we proceed by induction on $i = 0, 1, 2, \ldots, b$. The statement is clear for picture $\mathbf{P}_0$. In the induction step ($i \implies i+1$) we have to inspect the amalgamation of copies of $\mathbf{P}_i$ along the copies of systems $\mathbf{B}_i$ in $\mathbf{C}_{i+1}$. It is clear that

$\mathbf{P}_{i+1}$ belongs again to $\mathrm{Forb}_{L^\bullet}(\mathcal{T})$ (as the forbidden substructures in $\mathcal{T}$ are all complete). It remains to show that $\approx_{\mathbf{C}}$ is a star equivalence of $\mathbf{C}$. Because $\mathbf{A}$ is assumed to be a good structure and because $\mathbf{B}_i$ have every vertex in a copy of $\mathbf{A}$, we know that every star with leaf in $\mathbf{B}_i$ also contains centre in $\mathbf{B}_i$. By Lemma 6.1 we a get star equivalence on $\mathbf{C}_{i+1}$. The star equivalence is preserved by the free amalgamations of $\mathbf{P}_i$ over $\mathbf{C}_{i+1}$ because every time we unify leaves of a star we also unify the centre. Consequently $\mathbf{P}_{i+1}$ has star equivalences.

□

7 Putting it together: Bowtie-free graphs have a Ramsey lift

We already introduced expansion of good graphs in Section 3. Recall that for a given ordered good bowtie-free graph $\mathbf{G}$ the $L_2(\mathbf{G})$ denotes the lift graph which is our ordered expansion of $\mathbf{G}$. Lifts $L_2(\mathbf{G})$ are alternatively described as complete structures with forbidden (coloured) digons and triangles (Theorem 3.4).

In this section we prove main Theorem 5.2:

Proof. Clearly we may assume that $\mathbf{B}$ is a good L_2-structure ($\mathbf{B} \in \mathcal{B}$), i.e. $\mathbf{B}$ is the lift $L_2(\mathbf{G_B})$ of an ordered good (bowtie-free) graph $\mathbf{G_B} \in \mathcal{B}_0$.

For $\mathbf{A} \in \overline{\mathcal{B}}$ there exists, up to isomorphism, unique minimal good L_2-structure $\widehat{\mathbf{A}}$ such that $\mathbf{A}$ is a substructure of $\widehat{\mathbf{A}}$. This just means that we complete a centre $c(\mathbf{A})$ to a "full" centre by adding to every vertex left-, or right-vertex or completing some vertices to K_4. This correspondence $\mathbf{A} \to \widehat{\mathbf{A}}$ is functorial in the sense that the correspondence of $\binom{\mathbf{B}}{\mathbf{A}}$ and $\binom{\mathbf{B}}{\widehat{\mathbf{A}}}$ is one to one. Thus we may assume that both $\mathbf{A}$ and $\mathbf{B}$ are good L_2-structure (in the sense of Definition 3.4).

We consider $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ as reduced structures of $\mathbf{A}$ and $\mathbf{B}$ in $\mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$. By Theorem 6.2 there exists $\mathbf{D}^\bullet \in \mathrm{Forb}^{\approx}_{L^\bullet}(\mathcal{T})$ satisfying

$$\mathbf{D}^\bullet \longrightarrow (\mathbf{B}^\bullet)^{\mathbf{A}^\bullet}_{t=2}.$$

We now produce $\mathbf{E} \in \mathcal{B} \subseteq \overline{\mathcal{B}}$ which is a completion of $\mathbf{D}^\bullet$ that contains $\mathbf{D}^\bullet$ as an non-induced substructure in a way that every copy of $\mathbf{B}^\bullet$ in $\mathbf{D}$ can be extended to induced copy of $\mathbf{B}$ in $\mathbf{E}$.

Without loss of generality we assume that all vertices and edges of $\mathbf{D}^\bullet$ are contained in a copy of $\mathbf{B}^\bullet$. $\mathbf{E}$ will be constructed iteratively. We start by putting $\mathbf{E} = \mathbf{D}^\bullet$ and extend its vertex sets and relations as follows:

1. **Add missing vertices of vertex centres:**

   $\mathbf{D}^\bullet$ is a reduced structure. We now reverse this operation. For every central vertex $u \in R^L_{\mathbf{E}}$ we introduce new vertex $v \in R^R_{\mathbf{E}}$. We also add an edge of type 0 in between u and v. Similarly for every vertex $u_1 \in R^1_{\mathbf{E}}$ we introduce new vertices $v_2 \in R^2_{\mathbf{E}}$, $v_3 \in R^3_{\mathbf{E}}$, $v_4 \in R^4_{\mathbf{E}}$ and type 0 edges forming an K_4 on v_1, v_2, v_3, v_4.

2. **Complete the centre of E**:

   Consider a pair of central vertices $u, v \in E$ that are not connected by an L_2-edge. Denote by u' and v' the vertex in $R^L_{\mathbf{E}}$ or $R^1_{\mathbf{E}}$ belonging to the same vertex centre as u and v respectively. We add an L_2-edge representing the isomorphism type of the two vertex centres connected by edges as described by L_2-edge in between u' and v' if it exist or disjoint otherwise.

3. **Complete the non-central part E**:

   For every pair of vertices u, v that are not connected by an L_2-edge we determine the isomorphism type based on L_2-edges in between C_u and C_v (where C_u is the centre of u and C_v is the centre of v) and type 1 edges in between $u \cup C_u$ and $v \cup C_v$.

The procedure did not affect L_0-reduct of $\mathbf{E}$. Obviously $\mathbf{E}$ is complete. It remains to verify that $\mathbf{E} \in \mathrm{Forb}_{L_2}(\mathcal{T})$. (By Theorem 3.4 L_2 consists of forbidden triangles and digons.) The only bad triangles in $\mathbf{E}$ may be ones consisting of one or more newly introduced edges.

First observe that the central part of $\mathbf{E}$ is consistent, since the edges added represent precisely the isomorphism types of the L_0-reduct of $\mathbf{E}$ and only forbidden substructures in L_0-reduct are forbidden triangles.

The same observation do not extend to non-central vertices: the chimneys are not described by forbidden triangles in L_0-reducts. We however make use of the star equivalence. In $\mathbf{E}$ the star equivalence and chimney equivalence coincide and thus the chimneys are well formed.

The last part that remains to be verified is the consistency of L_2-edges in between non-central vertices (and non-central and central vertices). The edges introduced by completion process were created exactly as in the construction of L_2-lifts. It remains to observe that they are consistent also with the L_2-edges in $\mathbf{D}$. This follows from our assumption that every edge is part of copy of $\mathbf{A}$ and $\mathbf{A}$ is a good structure and their vertex centres are thus already given by $\mathbf{A}$. □

8 Concluding remarks

The techniques presented in this paper can be carried to other classes of structures that are defined by forbidden monomorphisms. More generally, the centre of a structure is defined then by the algebraic closure, studied e.g. in [6].

Bowtie-free graphs motivate a strategy to establish the Ramseyness of a class in the following basic steps:

1. Show that structures have free amalgamation over closed sets (Lemma 3.1).

2. Identify different types of vertices in the universal graphs.

3. Introduce ordering of structures that is convex on individual types of vertices.

4. Construct a lift that for every finite substructure uniquely describe the isomorphism type of its algebraic closure. If the algebraic closure is unary (that is the algebraic closure of set is the union of algebraic closures of its vertices) and finite, the lift will use finitely many new relations of finite arity bounded by maximal arity of the shadow.

5. From the unarity of closure operator show that closed sets are preserved thorough the Partite Construction.

It is conjectured in [5] that for classes defined by forbidden monomorphisms from one forbidden graph the algebraic closure operator is either unary or there so no universal ω-categorical graph at all. We believe that all such classes with unary closure operator can be proved to be Ramsey by a generalisation of a proof presented here. On the other hand a simple example is given in [5] showing that the closure however does not need to be unary for classes defined by forbidden homomorphism from more than one connected graph. Our techniques does not seem to directly generalise for this case.

Other important case is the situation where the amalgamation is not free over closed sets. Several such classes with strong amalgamation have been proved to be Ramsey by means of partite construction (among those the classes mentioned in the introduction: partial orders, metric spaces and classes $\mathrm{Forb}_H(\mathcal{F})$).

We hope that it is possible to combine both techniques to obtain Ramsey results on even more restricted classes of graphs.

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