Strongly Polynomial Sequences as Interpretations

A.J. Goodall *

Computer Science Institute of Charles University (IUUK and ITI)

Malostranské nám.25, 11800 Praha 1, Czech Republic

andrew@iuuk.mff.cuni.cz

J. Nešetřil †

Computer Science Institute of Charles University (IUUK and ITI)

Malostranské nám.25, 11800 Praha 1, Czech Republic

nesetril@iuuk.mff.cuni.cz

P. Ossona de Mendez ‡

Centre d'Analyse et de Mathématiques Sociales (CNRS, UMR 8557)

190-198 avenue de France, 75013 Paris, France

and

Computer Science Institute of Charles University (IUUK)

Malostranské nám.25, 11800 Praha 1, Czech Republic

pom@ehess.fr

December 15, 2015

Abstract

A strongly polynomial sequence of graphs (G_n) is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs such that, for every graph F, the number of homomorphisms from F to G_n is a fixed polynomial function of n (depending on F). For example, (K_n) is strongly polynomial since the number of homomorphisms from F to K_n is the chromatic polynomial of F evaluated at n. In earlier work of de la Harpe and Jaeger, and more recently of Averbouch, Garijo, Godlin, Goodall, Makowsky, Nešetřil, Tittmann, Zilber and others, various examples of strongly polynomial sequences and constructions for families of such sequences have been found, leading to analogues of the chromatic polynomial for fractional colourings and acyclic colourings, to choose two interesting examples.

We give a new model-theoretic method of constructing strongly polynomial sequences of graphs that uses interpretation schemes of graphs in more general relational structures. This surprisingly easy yet general method encompasses all previous constructions and produces many more. We conjecture that, under mild assumptions, all strongly polynomial sequences of graphs can be produced by the general method of quantifier-free

*Supported by grant ERCCZ LL-1201 of the Czech Ministry of Education

†Supported by grant ERCCZ LL-1201 of the Czech Ministry of Education, CE-ITI P202/12/G061 of GAČR, and LIA STRUCO

‡Supported by grant ERCCZ LL-1201 of the Czech Ministry of Education and LIA STRUCO, and partially supported by ANR project Stint under reference ANR-13-BS02-0007

interpretation of graphs in certain basic relational structures (essentially disjoint unions of transitive tournaments with added unary relations). We verify this conjecture for strongly polynomial sequences of graphs with uniformly bounded degree.

Keywords: graph homomorphism, graph polynomial, relational structure, interpretation scheme
Subject classification (MSC 2010) Primary: 05C31, 05C60 Secondary: 03C13, 03C98

1 Introduction

The chromatic polynomial $P(G, x)$ of a graph G, introduced by Birkhoff over a century ago, is such that for a positive integer n the value $P(G, n)$ is equal to the number of proper n-colorings of the graph G. Equivalently, $P(G, n)$ is the number $\hom(G, K_n)$ of homomorphisms from G to the complete graph K_n. It can thus be considered that the sequence $(K_n)_{n\in\mathbb{N}}$ defines the chromatic polynomial by means of homomorphism counting.

A *strongly polynomial sequence* of graphs is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs such that, for every graph F, the number of homomorphisms from F to G_n is a polynomial function of n (the polynomial depending on F and the sequence $(G_n)_{n\in\mathbb{N}}$, but not on n). A sequence $(G_n)_{n\in\mathbb{N}}$ of finite graphs is *polynomial* if this condition holds for sufficiently large $n \geq n_F$. The sequence of complete graphs (K_n) provides a classical example of a strongly polynomial sequence. A homomorphism from a graph F to a graph G is often called a *G-colouring of F*, the vertices of G being the "colours" assigned to vertices of F and the edges of G specifying the allowed colour combinations on the endpoints of an edge of F.

The notion of (strongly) polynomial sequences of graphs was introduced by de la Harpe and Jaeger [9] (as a generalization of the chromatic polynomial), in a paper which includes a characterization of polynomial sequences of graphs via (induced) subgraph counting and the construction of polynomial sequences by graph composition. The notion of a (strongly) polynomial sequence extends naturally to relational structures, thus allowing the use of standard yet powerful tools from model theory, like interpretations [10, 12]. We use this to provide a construction which encompasses all previous constructions of strongly polynomial sequences and produces many more.

The "generalized colourings" introduced in [15, 11] include only colourings invariant under all permutations of colours, which holds for K_n-colourings (that is, proper n-colourings), but not in general for G_n-colourings for other sequences of graphs $(G_n)_{n\in\mathbb{N}}$. Makowsky [14] moves towards a classification of polynomial graph invariants, but one that does not include the class of invariants we define in this paper. On the other hand, generalized colourings in the sense defined in [15, 11] do include harmonious colourings (proper colourings with the further restriction that a given pair of colours appears only once on an edge) and others not expressible as the number of homomorphisms to terms of a graph sequence. Nevertheless, the formalism we introduce in this paper also extends to these types of colourings. We show that strongly polynomial sequences $(G_n)_{n\in\mathbb{N}}$ in the sense of de la Harpe and Jaeger (number of homomorphisms to G_n polynomial in n) have the further property that when imposing any condition on mappings from G to G_n that is expressible by a quantifier-free formula (such as being a homomorphism), the number of such mappings is again a polynomial in n (dependent only on the quantifier-free formula and on G). From this it is immediately seen that harmonious colourings and acyclic colourings, for example, are counted by polynomial functions just like the chromatic polynomial for usual proper colourings. (See Proposition 2.9, and its Corollary 2.10 and the paragraph that follows it.)

Garijo, Goodall and Nešetřil [7] give a construction of a broad class of strongly polynomial sequences by using representations of graphs by coloured rooted trees, which in particular incorporates the Tittmann–Averbouch–Makowsky polynomial [16] (not obtainable by graph composition and other operations known from [9] for building new polynomial sequences from old). In the language of this paper, this representation of graphs is an intepretation of graphs in coloured rooted trees and we thus find that the construction of [7] is a special instance of our method (see Section 5.1.6 below).

We extend the scope of the term "strongly polynomial" to sequences of general relational structures. The property of a sequence of relational structures being strongly polynomial is preserved under a rich variety of transformations afforded by the model-theoretic notion of an interpretation scheme. We start with "trivially" strongly polynomial sequences of relational structures, made from basic building blocks, and then by interpretation project these sequences onto graph sequences that are also strongly polynomial. The interpretation schemes that can be used here are wide-ranging (they need only be quantifier-free in their specification), and therein lies the power of the method. All constructions of strongly polynomial sequences that have been devised in [9] and [7] are particular cases of such interpretation schemes for graph structures. Indeed, we conclude the paper with the conjecture that (under mild assumptions) *all* strongly polynomial sequences of graphs might be produced by the schema we describe here. In Section 5.2 this is verified for the case of sequences of graphs with uniformly bounded degree.

2 Preliminaries

2.1 Relational structures

A *relational structure* $\mathbf{A}$ with *signature* λ is defined by its *domain* A, a set whose elements we shall call vertices, and relations with names and arities as defined in λ. A relational structure will be denoted by an uppercase letter in boldface and its underlying domain by the corresponding lightface letter; for brevity we refer to a relational structure $\mathbf{A}$ with signature λ as a λ-*structure*, and may just give the *type* (list of arities given by λ) when the symbols used for the corresponding relations are not of importance. A 1-ary relation defines a subset of the domain and will be called a *label*, or a *mark* (a special type of labelling defined at the end of this section). A 2-ary relation defines edges of a digraph on vertex set the domain, and a graph when the relation is symmetric. When the signature λ contains only arities 1 and 2 we have a digraph together with labels on edges and vertices: relations of arity 1 are labels on vertices (where in general a vertex may receive more than one label) and relations of arity 2 are labelled edges (two vertices may be joined by edges of different labels).

The symbols of the relations and constants defined in λ define the non-logical symbols of the first-order language $\mathrm{FO}(\lambda)$ associated with λ-structures. We take first-order logic with equality as a primitive logical symbol and which is always interpreted as standard equality, so the equality relation does not appear in the signature λ. In what follows λ will be finite, in which case $\mathrm{FO}(\lambda)$ is countable. The variable symbols will be taken from the set $\{x_i : i \in \mathbb{N}\}$ or $\{y_i : i \in \mathbb{N}\}$, or, when double indexing is convenient, from $\{x_{i,j} : i, j \in \mathbb{N}\}$. The subset of $\mathrm{FO}(\lambda)$ consisting of formulas with exactly p free variables is denoted by $\mathrm{FO}_p(\lambda)$. The fragment of $\mathrm{FO}(\lambda)$ consisting of quantifier-free formulas is denoted by $\mathrm{QF}(\lambda)$, and $\mathrm{QF}_p(\lambda)$ denotes those quantifier-free formulas with exactly p free variables.

For a formula $\phi \in \mathrm{FO}_p(\lambda)$ and a λ-structure $\mathbf{A}$ we define the *satisfaction set*

$$\phi(\mathbf{A}) = \{(v_1, \dots, v_p) \in A^p : \ \mathbf{A} \models \phi(v_1, \dots, v_p)\},$$

where $\phi(v_1, \dots, v_p)$ is the formula obtained upon substituting v_i for each free variable x_i of ϕ, $i = 1, \dots, p$. A *homomorphism* from a λ-structure $\mathbf{A}$ to a λ-structure $\mathbf{B}$ is a mapping $f : A \to B$ that preserves relations, that is, which has the property that for each relation R in λ of any given arity r it is the case that $R(f(v_1), \dots, f(v_r))$ in $\mathbf{B}$ whenever $R(v_1, \dots, v_r)$ in $\mathbf{A}$. When $\mathbf{A}$ is a graph and R the relation representing adjacency of vertices this is a graph homomorphism as usually defined.

The number of homomorphisms from $\mathbf{A}$ to $\mathbf{B}$ is denoted by $\hom(\mathbf{A}, \mathbf{B})$.

A κ-structure $\mathbf{A}$ and a λ-structure $\mathbf{B}$ are *weakly isomorphic*, denoted by $\mathbf{A} \approx \mathbf{B}$, if there exists a bijection t between the symbols in κ and the symbols in λ and a bijection $f : A \to B$ such that, for every $R \in \kappa$, the relations R and $t(R)$ have the same arity (here denoted by r) and for every $v_1, \dots, v_r \in A$ we have

$$\mathbf{A} \models R(v_1, \dots, v_r) \quad \iff \quad \mathbf{B} \models t(R)(f(v_1), \dots, f(v_r)).$$

In other words, $\mathbf{A} \approx \mathbf{B}$ if $\mathbf{A}$ and $\mathbf{B}$ are the same structure, up to renaming of the relations and relabelling of the vertices.

For signatures κ and λ, we denote by $\kappa \sqcup \lambda$ the signature obtained from the disjoint union of κ and λ. The *strong sum* $\mathbf{A} \oplus \mathbf{B}$ of a κ-structure $\mathbf{A}$ and a λ-structure $\mathbf{B}$ is the $\kappa \sqcup \lambda$-structure whose domain is the disjoint union $A \sqcup B$ of the domains of $\mathbf{A}$ and $\mathbf{B}$, where for every $R \in \kappa$ and $S \in \lambda$ (with respective arities r and s) and for every $v_1, \dots, v_{\max(r,s)}$ in $A \sqcup B$ it holds that

$$\begin{aligned}
\mathbf{A} \oplus \mathbf{B} \models R(v_1, \dots, v_r) \quad &\iff \quad (v_1, \dots, v_r) \in A^r \text{ and } \mathbf{A} \models R(v_1, \dots, v_r), \\
\mathbf{A} \oplus \mathbf{B} \models S(v_1, \dots, v_s) \quad &\iff \quad (v_1, \dots, v_s) \in B^s \text{ and } \mathbf{S} \models S(v_1, \dots, v_s).
\end{aligned}$$

Note that the strong sum is not commutative, but we do have

$$\mathbf{A} \oplus \mathbf{B} \approx \mathbf{B} \oplus \mathbf{A}.$$

A class $\mathcal{C}$ of λ-structures is *marked* by a relation $U \in \lambda$ if U is unary, and for every $\mathbf{A} \in \mathcal{C}$ we have

$$\mathbf{A} \models (\forall x)\ U(x).$$

2.2 Sequences of relational structures

We start by defining the notion that is the subject of this paper.

Definition 2.1. A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of λ-structures is *strongly polynomial* if for every quantifier-free formula ϕ there is a polynomial P_ϕ such that $|\phi(\mathbf{A}_n)| = P_\phi(n)$ holds for every $n \in \mathbb{N}$.

Remark 2.2. A sequence is *polynomial* if for every quantifier-free formula ϕ there is a polynomial P_ϕ and an integer n_ϕ such that $|\phi(\mathbf{A}_n)| = P_\phi(n)$ holds for every integer $n \geq n_\phi$. We shall only consider strongly polynomial sequences in this paper, but occasionally it will help to clarify what is involved in the property of being a strongly polynomial sequence by giving examples of sequences that are polynomial but not strongly polynomial.

First we shall formulate equivalent criteria for a sequence of structures to be strongly polynomial in terms of homomorphisms or induced substructures (Theorem 2.5 below). For graph structures this will make a direct connection to the notion as originally defined by de la Harpe and Jaeger [9].

For this we require the following lemma:

Lemma 2.3. *Let λ be a signature for relational structures. For every formula ϕ in* QF(λ)*, there exist λ-structures $\mathbf{F}_1, \ldots, \mathbf{F}_\ell$ and integers $c_1, \ldots, c_\ell$ such that for every λ-structure $\mathbf{A}$ we have*

$$|\phi(\mathbf{A})| = \sum_i c_i \hom(\mathbf{F}_i, \mathbf{A}).$$

Proof. Let $\phi \in \mathrm{QF}(\lambda)$ be quantifier-free with free variables $x_1, \ldots, x_p$.

We first put ϕ in disjunctive normal form, with basic terms $(x_i = x_j)$ (for $1 \leq i < j \leq p$) and $R(x_{i_1}, \ldots, x_{i_r})$ for relation R in λ with arity r and $i_1, \ldots, i_r$ in $\{1, \ldots, p\}$. Thus ϕ is logically equivalent to

$$\bigvee_{\mathcal{P}} \zeta_{\mathcal{P}} \wedge \phi_{\mathcal{P}}, \tag{1}$$

where the disjunction runs over partitions $\mathcal{P}$ of $\{1, \ldots, p\}$, where $\zeta_{\mathcal{P}}$ is the conjunction of all equalities and non-equalities that have to hold between free variables $x_1, \ldots, x_p$ in order that $\mathcal{P}$ induces the partition of free variables into their k ($1 \leq k \leq p$) equality classes, and where $\phi_{\mathcal{P}}$ is a formula with k free variables defining the λ-structure $\mathbf{F}_{\mathcal{P}}$ induced by $x_{i_1}, \ldots, x_{i_k}$ for arbitrary choice of representatives $i_1, \ldots, i_k$ of the parts of $\mathcal{P}$. As all the terms in (1) are mutually exclusive, we have

$$|\phi(\mathbf{A})| = \sum_{\mathcal{P}} \mathrm{ind}(\mathbf{F}_{\mathcal{P}}, \mathbf{A}),$$

where $\mathrm{ind}(\mathbf{F}, \mathbf{A})$ denotes the number of injective mappings $f : F \to A$ defining an isomorphism between $\mathbf{F}$ and its image.

We wish to rewrite this sum in terms of induced substructures as one in terms of homomorphism numbers, and we achieve this in two steps. First we move from counting induced substructures to counting injective homomorphisms,

$$\mathrm{inj}(\mathbf{F}, \mathbf{A}) = \sum_{\substack{\mathbf{F}' : F' = F \\ \forall R \in \lambda\ R(\mathbf{F}') \supseteq R(\mathbf{F})}} \mathrm{ind}(\mathbf{F}', \mathbf{A}),$$

in which $\mathrm{inj}(\mathbf{F}, \mathbf{A})$ denotes the number of injective homomorphisms from $\mathbf{F}$ into $\mathbf{A}$ and $R(\mathbf{F}) = \{(v_1, \ldots, v_r) \in F^r : R(v_1, \ldots, v_r)\}$ is the set of tuples satisfying the r-ary relation R in $\mathbf{F}$, and similarly $R(\mathbf{F}')$ denotes those tuples satisfying the relation R in $\mathbf{F}'$. From this identity, by inclusion-exclusion we obtain

$$\mathrm{ind}(\mathbf{F},\mathbf{A}) = \sum_{\substack{\mathbf{F}':F'=F\\ \forall R\in\lambda\ R(\mathbf{F}')\supseteq R(\mathbf{F})}} \left(\prod_{R\in\lambda}(-1)^{|R(\mathbf{F})|-|R(\mathbf{F}')|}\right)\mathrm{inj}(\mathbf{F}',\mathbf{A}).$$

The second step is to move from counting injective homomorphisms to counting all homomorphisms, the relationship between which is given by

$$\mathrm{hom}(\mathbf{F},\mathbf{A}) = \sum_{\Theta}\mathrm{inj}(\mathbf{F}/\Theta,\mathbf{A}),$$

where the sum is over partitions Θ of the domain F of $\mathbf{F}$ and $\mathbf{F}/\Theta$ is the structure obtained from $\mathbf{F}$ by identifying elements of its domain F that lie in the same block of Θ. We then obtain

$$\mathrm{inj}(\mathbf{F},\mathbf{A}) = \sum_{\Theta}\mu(\Theta)\mathrm{hom}(\mathbf{F}/\Theta,\mathbf{A}),$$

where

$$\mu(\Theta) = \prod_{I\in\Theta}(-1)^{|I|-1}(|I|-1)!$$

is the Möbius function of the lattice of partitions of F. The statement of the lemma now follows. □

Remark 2.4. In the context of graphs (structures with signature comprising a symmetric binary relation) the identities used in the proof of Lemma 2.3 between counts of induced subgraphs, homomorphisms and injective homomorphisms find widespread application (see for example [4]).

We now come to the promised reformulation of the notion of a strongly polynomial sequence of structures.

Theorem 2.5. *The following are equivalent for a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of λ-structures:*

(i) The sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is strongly polynomial;

(ii) For each quantifier-free formula ϕ there is a polynomial P_ϕ such that $|\phi(\mathbf{A}_n)| = P_\phi(n)$ for each $n\in\mathbb{N}$;

(iii) For each λ-structure $\mathbf{F}$ there is a polynomial $P_{\mathbf{F}}$ such that $\mathrm{hom}(\mathbf{F},\mathbf{A}_n) = P_{\mathbf{F}}(n)$ *for each $n\in\mathbb{N}$;*

(iv) For each λ-structure $\mathbf{G}$ there is a polynomial $P_{\mathbf{G}}$ such that $\mathrm{ind}(\mathbf{G},\mathbf{A}_n) = P_{\mathbf{G}}(n)$ *for each $n\in\mathbb{N}$.*

Proof. Items (i) and (ii) are equivalent by definition. As homomorphisms and finite induced substructures can be expressed in QF, items (iii) and (iv) are both special cases of (ii). Finally, Lemma 2.3 shows that (iii) implies (ii), and the proof of the same lemma that (iv) also implies (ii). □

Remark 2.6. In Theorem 2.5, the equivalence also holds with weaker conditions in which the existence of a polynomial function is replaced by the existence of a rational function. Indeed, assume $|\phi(\mathbf{A}_n)| = f(x) = P(x)/Q(x)$ is a rational function, where P and Q are polynomials. Then there exist polynomials $R(x), S(x)$ such that $f(x) = S(x) + R(x)/Q(x)$ and $\deg R < \deg Q$. For sufficiently large n, it follows that $-1 < R(x)/Q(x) < 1$. As $f(x)$ takes only integral values on integers, it follows that $R(n)/Q(n) = 0$ for sufficiently large n. Hence $R = 0$ and f is a polynomial function.

Remark 2.7. Assume κ, λ are signatures such that κ is a subset of λ. Then every κ-structure can be considered as a λ-structure. The notion of strongly polynomial sequence is robust in the sense that a sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of κ-structures is strongly polynomial if and only if $(\mathbf{A}_n)_{n\in\mathbb{N}}$ (considered as a sequence of λ-structures) is strongly polynomial: indeed, for every λ-structure $\mathbf{F}$, either $\mathbf{F}$ contains a relation not in κ and thus $\hom(\mathbf{F}, \mathbf{A}_n) = 0$ for every $n \in \mathbb{N}$, or $\mathbf{F}$ can be considered as a κ-structure, and the number of homomorphisms from $\mathbf{F}$ to $\mathbf{A}_n$ does not depend on the signature considered.

Remark 2.8. Consider graph structures, with adjacency relation denoted by $\sim$. For a fixed graph G on vertex set $[k]$, if

$$\phi_G(v_1, \dots, v_k) = \bigwedge_{ij\in E(G)} (v_i \sim v_j), \tag{2}$$

then the satisfaction set of ϕ_G for a given graph structure H is

$$\phi_G(H) = \{(v_1, \dots, v_k) : i \mapsto v_i \text{ is a homomorphism } G \to H\}.$$

Thus $|\phi_G(H)| = \hom(G, H)$ and by Theorem 2.5 a sequence of graphs (G_n) is strongly polynomial if and only if for each graph G there is some polynomial P_G such that $\hom(G, G_n) = P_G(n)$ for all $n \in \mathbb{N}$. This is how strongly polynomial sequences of graphs are defined in [9].

For graphical structures the perhaps surprising equivalence of (ii) and (iii) in Theorem 2.5 has the following consequence as a special case. We isolate it for its application to certain well-studied types of colourings (see Corollary 2.10 below):

Proposition 2.9. *Let (G_n) be a strongly polynomial sequence of (not necessarily simple) graphs, $\mathcal{H}$ a family of (isomorphism types of) graphs, and G a graph.*

Then the number of homomorphisms $f : G \to G_n$ with the property that for each $v \in V(G_n)$ the inverse image $f^{-1}(\{v\})$ induces a subgraph of G which is isomorphic to a graph in $\mathcal{H}$ is a fixed polynomial function of n.

More generally, fixing $r \in \mathbb{N}$, the number of homomorphisms $f : G \to G_n$ with the property that for each $U \in \binom{V(G_n)}{r}$ the inverse image $f^{-1}(U)$ induces a subgraph of G which is isomorphic to a graph in $\mathcal{H}$ is a fixed polynomial function of n.

Proof. The result is immediate from the definition of a strongly polynomial sequence once we find a quantifier-free formula expressing the fact that a mapping $f : G \to G_n$ is not only a homomorphism but satisfies the given additional condition on inverse images. Since G is fixed and finite, the intersection of $\mathcal{H}$ with the collection of induced subgraph types of G is finite and we can indeed do this simply by taking an exhaustive finite disjunction of finite conjunctions.

To this end, let $V(G) = [k]$ and ϕ_G be defined as in (2), indicating that $i \mapsto v_i$ is a homomorphism from G to another graph (G_n, once we interpret the free variables $v_1, \ldots, v_k$ as ranging over $V(G_n)$). For a partition $\mathcal{P}$ of $[k]$ into subsets $I_1, \ldots, I_\ell$, $1 \leq \ell \leq |V(G_n)|$, let $\zeta_\mathcal{P}$ be the conjunction of all those equalities and non-equalities that need to hold between the free variables $v_1, \ldots, v_k$ in order that they are partitioned according to their indices by the corresponding blocks of $\mathcal{P}$.

For our fixed graph G on vertex set $[k]$ we now simply form the disjunction over all partitions $\mathcal{P}$ of $[k]$ such that each block induces a subgraph isomorphic to a graph that belongs to the finite set $\{H \in \mathcal{H} : |V(H)| \leq k\}$: the formula

$$\psi_{G,\mathcal{H}} = \phi_G \wedge \bigvee_{\substack{\mathcal{P}=\{I_1,\ldots,I_\ell\} \\ G[I_j]\cong H \in \mathcal{H}}} \zeta_\mathcal{P}$$

has satisfaction set $\psi_{G,\mathcal{H}}(G_n)$ equal to all homomorphisms $i \mapsto v_i$ from G to G_n with the property that the vertices of G mapped to a given vertex of G_n induce a subgraph of G isomorphic to a graph belonging to $\mathcal{H}$.

For the generalization to r-subsets, redfine $\psi_{G,\mathcal{H}}$ as

$$\psi_{G,\mathcal{H}} = \phi_G \wedge \bigvee_{\substack{\mathcal{P}=\{I_1,\ldots,I_\ell\} \\ G[I_{j_1}\cup\cdots\cup I_{j_r}]\cong H \in \mathcal{H}}} \zeta_\mathcal{P},$$

where the disjunction is over all partitions $\mathcal{P}$ of $[k]$ such that the union of any r blocks induces a subgraph of G isomorphic to a graph that belongs to $\{H \in \mathcal{H} : |V(H)| \leq k\}$. □

As an application of Proposition 2.9 we have the following result, expressed in the terminology of [11, Sect. 3.3]:

Corollary 2.10. *Harmonious n-colourings, connected n-colourings and $mmc(t)$ n-colourings (a colour class induces a subgraph of G whose components have size bounded by t) of a graph G are each counted by a polynomial function of n.*

Proof. For harmonious n-colourings take (G_n) in Proposition 2.9 to be the strongly polynomial sequence (K_n) and $\mathcal{H}$ to be all graphs with at most one edge and count homomorphisms $f : G \to K_n$ with the property that for each pair of distinct elements $u, v \in V(K_n)$ the inverse image $f^{-1}(\{u, v\})$ induces a subgraph of G that belongs to $\mathcal{H}$.

Now take $(G_n) = (K_n^\circ)$ to be the sequence of complete graphs with a loop on each vertex. For connected colourings take $\mathcal{H}$ to be all connected graphs, and for $mmc(t)$ n-colourings $\mathcal{H}$ to be all graphs of size at most t, and count functions $f : G \to K_n^\circ$ with the property that for each $v \in V(K_n^\circ)$ the inverse image $f^{-1}(\{v\})$ induces a subgraph of G that belongs to $\mathcal{H}$. □

For another example in this vein, take $(G_n) = (K_n)$ and $\mathcal{H}$ the set of all forests in order to deduce that the number of *acyclic n-colourings* (no two colour classes induce a subgraph containing a cycle) is a polynomial in n, and if instead we take $\mathcal{H}$ to be the set of forests whose connected components are stars then we have the number of *star n-colourings* of G and this is a polynomial in n. (These types of colourings and the associated acyclic chromatic number and star chromatic number were first introduced over forty years ago by Grünbaum [8].)

We end this section with a few statements on the invariance of strongly polynomial sequences with respect to various operations.

Lemma 2.11. *Let* $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ *be strongly polynomial sequences of* λ_i*-structures, for* $1 \le i \le k$*. Then the sequence* $(\bigoplus_{i=1}^{k} \mathbf{A}_{i,n})_{n\in\mathbb{N}}$ *is strongly polynomial.*

Proof. By Lemma 2.3 it is sufficient to check polynomiality of $\hom(\mathbf{F}, \bigoplus_{i=1}^{k} \mathbf{A}_{i,n})$. Let $\mathbf{F}_1, \dots, \mathbf{F}_\ell$ be the connected components of $\mathbf{F}$. Then the proof follows from the identity

$$\begin{aligned}\hom(\mathbf{F}, \bigoplus_{i=1}^{k} \mathbf{A}_{i,n}) &= \prod_{j=1}^{\ell} \hom(\mathbf{F}_j, \bigoplus_{i=1}^{k} \mathbf{A}_{i,n}) \\ &= \prod_{j=1}^{\ell} \sum_{i=1}^{k} \hom(\mathbf{F}_j, \mathbf{A}_{i,n}).\end{aligned}$$

(Note that in the last equality we consider each $\mathbf{A}_{i,n}$ as a $\bigsqcup_{i=1}^{k} \lambda_i$-structure, which it is safe to do according to Remark 2.7.) □

Lemma 2.12. *Let* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *be a strongly polynomial sequence of* λ*-structures and let* P *be a polynomial such that* $P(n) \in \mathbb{N}$ *for* $n \in \mathbb{N}$*. Then the sequence* $(\bigsqcup^{P(n)} \mathbf{A}_n)_{n\in\mathbb{N}}$ *is strongly polynomial, where* $\bigsqcup^{P(n)} \mathbf{A}_n$ *denotes the* λ*-structure obtained as the disjoint union of* $P(n)$ *copies of* $\mathbf{A}_n$*.*

Proof. By Lemma 2.3 it is sufficient to check polynomiality of $\hom(\mathbf{F}, \bigsqcup^{P(n)} \mathbf{A}_n)$. Let $\mathbf{F}_1, \dots, \mathbf{F}_\ell$ be the connected components of $\mathbf{F}$. Then the proof follows from the identity

$$\hom(\mathbf{F}, \bigsqcup^{P(n)} \mathbf{A}_n) = \prod_{j=1}^{\ell} \hom(\mathbf{F}_j, \bigsqcup^{P(n)} \mathbf{A}_n) = \prod_{j=1}^{\ell} (P(n) \hom(\mathbf{F}_j, \mathbf{A}_n)).$$

□

Lemma 2.13. *Let* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *be a strongly polynomial sequence of* λ*-structures and let* P *be a polynomial such that* $P(n) \in \mathbb{N}$ *for* $n \in \mathbb{N}$*. Then the sequence* $(\mathbf{A}_{P(n)})_{n\in\mathbb{N}}$ *is strongly polynomial.*

Proof. For every $\phi \in \mathrm{QF}(\lambda)$ there is a polynomial Q such that $|\phi(\mathbf{A}_n)| = Q(n)$ for each $n \ge 1$, as $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a strongly polynomial sequence. Thus $|\phi(\mathbf{A}_{P(n)})| = Q \circ P(n)$. It follows that the sequence $(\mathbf{A}_{P(n)})_{n\in\mathbb{N}}$ is strongly polynomial. □

3 Basic structures

Two special types of marked structures will be of particular interest in this paper:

- $\mathbf{E}$ is a structure whose domain is a singleton, and whose signature is a single unary relation U that satisfies $(\forall x)\ U(x)$;

- $\mathbf{T}_n$ is a transitive tournament of order n. Precisely, the domain of $\mathbf{T}_n$ is $[n] = \{1, \dots, n\}$ and its signature contains a single binary relation S with $\mathbf{T}_n \models S(i,j) \iff i < j$, and a single unary relation U, which satisfies $(\forall x)\ U(x)$.

The number of induced (U,S)-substructures of $\mathbf{T}_n$ isomorphic to a given (U,S)-structure $\mathbf{G}$ is $\binom{n}{r}$ if $\mathbf{G}$ is isomorphic to $\mathbf{T}_r$ for some positive integer r, and constantly zero otherwise. By Theorem 2.5, the sequence $(\mathbf{T}_n)$ is strongly polynomial.

Definition 3.1. A *basic structure with parameters* $(k, \ell, N_1, \dots, N_k)$ is a structure

$$\mathbf{B} = \overbrace{\mathbf{E} \oplus \cdots \oplus \mathbf{E}}^{\ell \text{ times}} \oplus \overbrace{\mathbf{T}_{N_1} \oplus \cdots \oplus \mathbf{T}_{N_k}}^{k \text{ times}}$$

which is the strong sum of ℓ marked vertices and k marked transitive tournaments of respective orders $N_1(\mathbf{B}), \dots, N_k(\mathbf{B})$. We denote by $\mathcal{B}_{k,\ell}$ the class of all basic structures with parameters $(k, \ell, N_1, \dots, N_k)$ for some positive integers $N_1, \dots, N_k$, and by $\beta_{k,\ell}$ the signature of these structures. It will be notationally convenient to assume that the relations in $\beta_{k,\ell}$ are $U_1^E, \dots, U_\ell^E$, $U_1^T, \dots, U_k^T$, $S_1, \dots, S_k$.

A *basic sequence* is a sequence $(\mathbf{B}_n)_{n \in \mathbb{N}}$ of basic structures $\mathbf{B}_n \in \mathcal{B}_{k,\ell}$ (for some fixed $k, \ell \in \mathbb{N}$) such that there are non-constant polynomials Q_i, $1 \le i \le k$ with $Q_i(n) = N_i(\mathbf{B}_n)$ (for every $1 \le i \le k$ and $n \in \mathbb{N}$).

It follows directly from Lemmas 2.13 and 2.11 that every basic sequence is strongly polynomial.

4 Strongly polynomial sequences by interpretations

The basic building blocks we use for constructing strongly polynomial graph sequences are marked tournaments $(\mathbf{T}_{P(n)})$ on a polynomial number of vertices, and the constant sequence $(\mathbf{E})$ consisting of a single marked vertex. From these we can produce all the strongly polynomial sequences given in [9] and [7] and much more. (In Section 5.1 we give a large selection of examples of strongly polynomial graph sequences that exhibits their diversity.) To do this we need just two operations: strong sum and graphical interpretation of structures. The latter is a potent operation for it produces graph sequences from strongly polynomial sequences of λ-structures of arbitrary signature λ, while strong sum is an essential operation for gluing together separately constructed sequences, from which a sequence of larger structures can be made. (Note that one could equivalently consider disjoint union and QF-interpretation in place of strong sum and QF-interpretation.)

4.1 Interpretation schemes

We begin with the definition of an interpretation scheme that we shall require. For more background to the model-theoretic results of this section we refer to [10, 12].

Definition 4.1. Let κ, λ be signatures, where the signature λ has q relational symbols $R_1, \ldots, R_q$ with respective arities $r_1, \ldots, r_q$. An *interpretation scheme* I of λ-structures in κ-structures with exponent p is a tuple $I = (p, \rho_0, \ldots, \rho_q)$, where p is a positive integer, $\rho_0 \in \mathrm{FO}_p(\kappa)$, and $\rho_i \in \mathrm{FO}_{pr_i}(\kappa)$, for $1 \leq i \leq q$.

For κ-structure $\mathbf{A}$, we denote by $I(\mathbf{A})$ the λ-structure $\mathbf{B}$ with domain $B = \rho_0(\mathbf{A})$ and relations defined by

$$\mathbf{B} \models R_i(\mathbf{v}_1, \ldots, \mathbf{v}_{r_i}) \quad \Longleftrightarrow \quad \mathbf{A} \models \rho_i(\mathbf{v}_1, \ldots, \mathbf{v}_{r_i})$$

(for $1 \leq i \leq q$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{r_i} \in B$).

Definition 4.2. A *QF-interpretation scheme* is an interpretation scheme in which all the formulas $\rho_i, 0 \leq i \leq q$, used to define it in Definition 4.1 are quantifier-free.

Example 4.3. Let us consider two signatures κ and λ with $\kappa \subset \lambda$. Then the following transformations are easily (and almost trivially) checked to be definable by QF-interpretation schemes:

- *Lift*, the canonical injection of κ-structures into a λ-structures (same relations);
- *Forget*, the canonical projection of λ-structures onto κ-structures (filters out relations not in κ);
- *Merge*, which maps $\lambda \sqcup \lambda$-structures into λ-structures by merging similar relations (so that $\mathrm{Merge}(\mathbf{A} \oplus \mathbf{B}) = \mathbf{A} + \mathbf{B}$);
- *Mark*, which maps λ-structures into λ^+-structures (where λ^+ is the signature obtained by adding to λ a new unary relation U) by putting every element in the relation U.

Since our goal is to construct strongly polynomial sequences of graphs, we shall have a particular use for interpretation schemes of graph structures in κ-structures.

Definition 4.4. A *graphical interpretation scheme* I of κ-structures is a triple (p, ι, ρ), where p is a positive integer, $\iota \in \mathrm{FO}_p(\kappa)$, and $\rho \in \mathrm{FO}_{2p}(\kappa)$ is symmetric (that is, such that $\vdash \phi(x, y) \leftrightarrow \phi(y, x)$). For every κ-structure $\mathbf{A}$, the interpretation $I(\mathbf{A})$ has vertex set $V = \iota(\mathbf{A})$ and edge set

$$E = \{\{\mathbf{u}, \mathbf{v}\} \in V \times V : \ \mathbf{A} \models \rho(\mathbf{u}, \mathbf{v})\}.$$

We have already mentioned a graphical interpretation scheme of digraph structures: that which interprets an orientation of a graph as the underlying undirected graph simply by forgetting the edge directions (for example K_n from $\mathbf{T}_n$). Taking the complement of a graph G is a graphical interpretation scheme (of graph structures) with $p = 1$ in which we take $\iota = 1$ (constantly true), and $\rho(x, y) = \neg R(x, y)$, where $R(x, y)$ represents adjacency between x and y. The square of the graph G, joining vertices x and y when they are adjacent or share a common neighbour, is a graphical interpretation scheme (of graph structures) with $p = 1$, $\iota = 1$, and $\rho(x, y) = R(x, y) \vee (\exists z \ R(x, z) \wedge R(z, y))$ (this one requires a quantifier). The line graph of a simple undirected graph G can be realized indirectly: orient the edges G and use a graphical interpretation scheme of digraph structures with $p = 2$, by taking $\iota(x, y) = R(x, y)$, where R is the antisymmetric relation representing oriented edges of G, and $\rho(x_1, y_1, x_2, y_2) = [(x_1 = y_2) \vee (y_1 = y_2) \vee (x_1 = y_2) \vee (x_2 = y_1)] \wedge \neg[(x_1 = x_2) \wedge (y_1 = y_2)] \wedge \neg[(x_1 = y_2) \wedge (x_2 = y_1)]$. Compare also Remark 5.2 in Section 5.1.4.

The following standard result from model theory (see for example [12, Section 3.4]) underlies the key role interpretation will play in moving from one strongly polynomial sequence of structures to another.

Let $I = (p, \rho_0, \ldots, \rho_q)$ be an interpretation scheme of λ-structures in κ-structures. We inductively define the mapping M_I from terms in $\mathrm{FO}(\lambda)$ to terms in $\mathrm{FO}(\kappa)$ by:

- $M_I(x_i) = (x_{i,1}, \ldots, x_{i,p})$ for variable x_i;
- $M_I(R_i(t_1, \ldots, t_{r_i})) = \rho_i(M_I(t_1), \ldots, M_I(t_{r_i}))$;
- $M_I(\phi \vee \psi) = M_I(\phi) \vee M_I(\psi)$;
- $M_I(\phi \wedge \psi) = M_I(\phi) \wedge M_I(\psi)$;
- $M_I(\neg\phi) = \neg M_I(\phi)$;
- $M_I((\exists x)\ \phi) = (\exists x_1 \ldots \exists x_p)\ \bigwedge_{i=1}^{p} \rho_0(x_i) \wedge M_I(\phi)$;
- $M_I((\forall x)\ \phi) = (\forall x_1 \ldots \forall x_p)\ \bigwedge_{i=1}^{p} \rho_0(x_i) \to M_I(\phi)$.

Then we define the mapping $\tilde{I} : \mathrm{FO}(\lambda) \to \mathrm{FO}(\kappa)$ by

$$\tilde{I}(\phi) = \bigwedge_{i=1}^{k} \rho_0(x_i)\ \wedge\ M_I(\phi),$$

where $x_1, \ldots, x_k$ are the free variables of ϕ.

Note that if all the ρ_i are quantifier-free then $\tilde{I}$ maps quantifier-free formulas to quantifier-free formulas.

Lemma 4.5. *If I is an interpretation scheme of λ-structures in κ-structures then for every $\phi \in \mathrm{FO}_r(\lambda)$ and every κ-structure $\mathbf{A}$ we have*

$$\phi(I(\mathbf{A})) = \tilde{I}(\phi)(\mathbf{A}).$$

As a corollary of Lemma 4.5 we have

Theorem 4.6. *If $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a strongly polynomial sequence of κ-structures and if I is a QF-interpretation scheme of λ-structures in κ-structures, then $(I(\mathbf{A}_n))_{n\in\mathbb{N}}$ is a strongly polynomial sequence of λ-structures.*

Marked structures are useful as marking provides a way to track components of strong sums, thus allowing the combination of interpretation schemes on components, as shown by the next lemma.

Lemma 4.7. *Let I_i $(1 \leq i \leq k)$ be interpretation schemes of λ_i-structures in κ_i-structures with exponent p_i. Assume that each κ_i contains a unary relation U_i.*

Then there exists an interpretation scheme I of $\bigsqcup_{i=1}^k \lambda_i$-structures in $\bigsqcup_{i=1}^k \kappa_i$-structures with exponent $p = \max p_i$ such that, for every κ_i-structure $\mathbf{A}_i$ marked by U_i $(1 \leq i \leq k)$, we have

$$I(\bigoplus_{i=1}^{k} \mathbf{A}_i) = \bigoplus_{i=1}^{k} I_i(\mathbf{A}_i).$$

Moreover, if all the I_i's are QF-interpretation schemes then there exists such an I which is a QF-interpretation scheme.

Proof. For $1 \leq i \leq k$, let $I_i = (p_i, \rho_0^i, \dots, \rho_{q_i}^i)$, where ρ_j^i has $r_{i,j}p_i$ free variables $(1 \leq j \leq q_i)$. We define the interpretation scheme $I = (p, \rho_0, \rho_{i,j})$ $(1 \leq i \leq k, 1 \leq j \leq q_i)$ with exponent p of $\bigsqcup_{i=1}^k \lambda_i$-structures in $\bigsqcup_{i=1}^k \kappa_i$-structures as follows: the formula ρ_0 is

$$\rho_0 : \bigvee_{i=1}^{k} \Big(\bigwedge_{j=1}^{p_i} U_i(x_j) \wedge \bigwedge_{j=p_i}^{p} (x_j = x_p) \wedge \rho_0^i(x_1, \dots, x_{p_i}) \Big)$$

and the formula $\rho_{i,j}$ (with $r_{i,j}p$ free variables) is defined by

$$\rho_{i,j} : \bigwedge_{\ell=1}^{r_{i,j}p} U_i(x_\ell) \wedge \rho_j^i(x_1, \dots, x_{p_i}, x_{p+1}, \dots, x_{p+p_i}, \dots, x_{(r_{i,j}-1)p+1}, \dots, x_{(r_{i,j}-1)p+p_i}).$$

Then I obviously satisfies the requirements of the lemma statement. $\square$

QF-interpretation schemes of strong sums are instrumental in the construction of strongly polynomial sequences, as exemplified by the next result.

Corollary 4.8. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ and $(\mathbf{B}_n)_{n\in\mathbb{N}}$ be strongly polynomial sequences of graphs. Then $(\mathbf{A}_n + \mathbf{B}_n)_{n\in\mathbb{N}}$, $(\mathbf{A}_n \times \mathbf{B}_n)_{n\in\mathbb{N}}$, $(\mathbf{A}_n \Box \mathbf{B}_n)_{n\in\mathbb{N}}$, $(\mathbf{A}_n \boxtimes \mathbf{B}_n)_{n\in\mathbb{N}}$, and $(\mathbf{A}_n[\mathbf{B}_n])_{n\in\mathbb{N}}$ are strongly polynomial sequence of graphs (formed respectively by disjoint union, direct product, Cartesian product, strong product, and lexicographic product).*

Proof. This follows from Theorem 4.6 and Lemma 2.11, by noticing that all the constructions listed here are QF-interpretations of $\mathrm{Mark}(\mathbf{A}_n) \oplus \mathrm{Mark}(\mathbf{B}_n)$. $\square$

Many more constructions can be used to combine strongly polynomial sequences of structures by means of strong sum and QF-interpretations, of which the following is an example.

Example 4.9. Let t be a fixed odd integer, and let $(\mathbf{A}_{i,n})_{n\in\mathbb{N}}$ $(1 \leq i \leq t)$ be a strongly polynomial sequences of graphs. We define $\mathbf{B}_n$ as the graph with vertex set $A_{1,n} \times \cdots \times A_{t,n}$ where $(u_1, \dots, u_t)$ is adjacent to $(v_1, \dots, v_t)$ in $\mathbf{B}_n$ if there is a majority of $i \in \{1, \dots, t\}$ such that u_i is adjacent to v_i in $\mathbf{A}_{i,n}$. Then $(\mathbf{B}_n)_{n\in\mathbb{N}}$ is a strongly polynomial sequence.

5 Interpretations of basic sequences

As already noted in Section 3, every basic sequence is strongly polynomial. It follows from Theorem 4.6 that this is also the case for their QF-interpretations:

Corollary 5.1. *If* $(\mathbf{B}_n)$ *is a basic sequence and* I *is a QF-interpretation of* λ*-structures in* $\beta_{k,\ell}$*-structures, then* $(I(\mathbf{B}_n))$ *is a strongly polynomial sequence of* λ*-structures.*

The class $\mathcal{P}$ of sequences $(\mathbf{A}_n)_{n\in\mathbb{N}}$ that can be obtained by QF-interpretations of basic sequences is quite rich. In particular, it is closed under the following operations, which can be used to construct new strongly polynomial sequences from old ones:

- Extracting a subsequence $(\mathbf{A}_{P(n)})_{n\in\mathbb{N}}$, where P is a polynomial such that $P(n)\in\mathbb{N}$ for $n\in\mathbb{N}$ (as $(\mathbf{B}_{P(n)})_{n\in\mathbb{N}}$ is a basic sequence);
- Applying a QF-interpretation scheme (as the composition of two QF-interpretation schemes defines a QF-interpretation scheme);
- Strong sums (according to Lemma 4.7);
- Multiplying by a polynomial P such that $P(n)\in\mathbb{N}$ for $n\in\mathbb{N}$ (as $\bigsqcup^{P(n)}\mathbf{A}_n$ is a QF-interpretation of $\mathbf{T}_{P(n)}\oplus\mathbf{A}_n$).

5.1 Examples

5.1.1 Rook's graph

The $n\times n$ Rook's graph is defined as the Cartesian product $K_n\Box K_n$ of two complete graphs. Since (K_n) is a strongly polynomial sequence of graphs (defining the chromatic polynomial), by Corollary 4.8 the sequence $(K_n\Box K_n)$ is also strongly polynomial. The graph $K_n\Box K_n$ has n^2 vertices and n^3-n^2 edges. This example shows that in general for a strongly polynomial sequence $(H_n)_{n\in\mathbb{N}}$ the polynomial $\hom(G,H_n)$ need not be a polynomial in $|V(H_n)|$.

5.1.2 Cycles

Our second example is in fact a non-example, illustrating how interpretation schemes that are not quantifier-free can produce sequences that are not strongly polynomial from basic sequences.

The cycle C_n on n vertices may be obtained by the following (non-QF) interpretation scheme applied to the basic sequence $\mathbf{T}_n$:

$$\begin{aligned}\iota(x):&\quad 1\\ \rho(x,y):&\quad \rho'(x,y)\vee\rho'(y,x),\end{aligned}$$

where

$$\begin{aligned}\rho'(x,y)=&(S_1(x,y)\wedge\forall z[\neg S_1(x,z)\vee\neg S_1(z,y)])\vee\\ &(S_1(x,y)\wedge\forall z[(S(x,z)\wedge S(z,y))\vee(x=z)\vee(z=y)]).\end{aligned}$$

The sequence $(C_n)_{n\in\mathbb{N}}$ is not strongly polynomial (for example, $\hom(C_3,C_n)$ is zero except for $n=3$), but, as shown in [9, Ex. B.5], has the weaker property that there is a polynomial P_G such that $\hom(G,C_n)=P_G(n)$ for $n>|V(G)|$.

5.1.3 Crown graphs

We start from the basic sequence $\mathbf{A}_n = \mathbf{E} \oplus \mathbf{E} \oplus \mathbf{T}_n$. Consider the graphical interpretation scheme $I = (2, \iota, \rho)$, where

$$\begin{aligned} \iota(x_1, x_2) :&\quad U_1^T(x_1) \wedge \neg U_1^T(x_2) \\ \rho(x_1, x_2, y_1, y_2) :&\neg(x_1 = y_1) \wedge \neg(U_1^E(x_2) \leftrightarrow U_1^E(y_2)). \end{aligned}$$

Then the graph obtained is the *crown graph* S_n ($K_{n,n}$ minus a perfect matching), and it follows that $(S_n)_{n\in\mathbb{N}}$ is a strongly polynomial sequence. Similarly, for every integer k, the sequence $(\mathrm{KG}_{n,k})_{n\in\mathbb{N}}$ of the *Kneser graphs* is strongly polynomial. For a graph G, the minimum ratio n/k such that there is a homomorphism from G to the Kneser graph $\mathrm{KG}_{n,k}$ is the *fractional chromatic number* of G.

5.1.4 Generalized Johnson graphs

We start from the basic sequence $\mathbf{A}_n = \mathbf{T}_n$ and consider, for fixed integer k and subset $D \subseteq [k]$, the graphical interpretation scheme $I = (k, \iota, \rho)$, where

$$\begin{aligned} \iota(x_1, \dots, x_k) :& \bigwedge_{i=1}^{k-1} S_1(x_i, x_{i+1}) \\ \rho(x_1, \dots, x_k, y_1, \dots, y_k) :& \bigvee_{\substack{I,J\subseteq[k]\\ |I|=|J| \\ |I|\in D}} \Big(\bigwedge_{i\notin I, j\notin J} \neg(x_i = y_j) \wedge \bigwedge_{i\in I} \bigvee_{j\in J} (x_i = y_j) \Big) \end{aligned}$$

Then $I(\mathbf{A}_n)$ is the generalized Johnson graph $J_{n,k,D}$, which is the graph with vertices $\binom{[n]}{k}$ and where X and Y are adjacent whenever $|X \cap Y| \in D$. By Corollary 5.1 the sequence $(J_{n,k,D})_{n\in\mathbb{N}}$ is strongly polynomial (for fixed k and D). The graph $J_{n,k,\{0\}}$ is the Kneser graph $\mathrm{KG}_{n,k}$ and $J_{n,k,\{k-1\}}$ is the original Johnson graph, which has recently gained prominence in the context of the graph isomorphism problem [3]. The quantity $\hom(G, J_{n,k,D})/\binom{n}{k}^{c(G)}$, where $c(G)$ is the number of connected components of G, is shown in [9] to depend only on the underlying cycle matroid of G, a property shared by the chromatic polynomial (the case $k = 1, D = \{0\}$) and the Tutte polynomial.

Remark 5.2. Similarly, let k be an integer and let $D \subseteq [k]$, and let $(G_n)_{n\in\mathbb{N}}$ be a strongly polynomial sequence of graphs. For $n \in \mathbb{N}$ define H_n as the graph whose vertices are the k-cliques of G_n, where two k-cliques of G_n are adjacent in H_n if the cardinality of their intersection belongs to D. Then the sequence of graphs $(H_n)_{n\in\mathbb{N}}$ is strongly polynomial. In particular, the sequence $(L(G_n))_{n\in\mathbb{N}}$ of line graphs is strongly polynomial.

5.1.5 Vertex-blowing of a fixed graph

Let F be a fixed graph with vertex set $[k]$. To each vertex i of F is associated a polynomial P_i such that $P_i(n) \in \mathbb{N}$ for $n \in \mathbb{N}$. Let $\mathbf{A}_n = \bigoplus_{i=1}^k \mathbf{T}_{P_i(n)}$. We define the graphical interpretation scheme $I = (1, \iota, \rho)$ by

$$\begin{aligned} \iota(x) :& \quad 1 \\ \rho(x,y) :& \bigvee_{ij \in E(F)} U_i^T(x) \wedge U_j^T(y) \end{aligned}$$

Then $I(\mathbf{A}_n)$ is the vertex-blowing of F, in which vertex i is replaced by $P_i(n)$ twin copies of i, and by Corollary 5.1 the sequence $(I(\mathbf{A}_n))_{n\in\mathbb{N}}$ is strongly polynomial.

5.1.6 Tree-blowing of a fixed rooted tree

Let F be a rooted tree with edge set $E = \{2, \dots, k\}$. To each edge $e \in E$ is associated a polynomial P_e such that $P_e(n) \in \mathbb{N}$ for $n \in \mathbb{N}$ and let P_1 be another such polynomial (for the root). Let $\mathbf{A}_n = \bigoplus_{i=1}^k \mathbf{T}_{P_i(n)}$. An F-path will be a sequence $(1, e_1, \dots, e_i)$ corresponding to a path from the root of F. Define the graphical interpretation scheme $I = (k, \iota, \rho)$ by

$$\begin{aligned} \iota(x_1, \dots, x_k) :& \bigvee_{F\text{-path } (a_1,\dots,a_t)} \left(\bigwedge_{i=1}^{t} U_{a_i}^T(x_i) \wedge \bigwedge_{i=t+1}^{k} (x_i = x_t) \right) \\ \rho(x_1, \dots, x_k, y_1, \dots, y_k) :& \ \rho'(x_1, \dots, x_k, y_1, \dots, y_k) \vee \rho'(y_1, \dots, y_k, x_1, \dots, x_k) \end{aligned}$$

where

$$\begin{aligned} &\rho'(x_1, \dots, x_k, y_1, \dots, y_k) : \\ &\qquad \bigvee_{i=1}^{k-1} \Big(\bigwedge_{j=1}^{i} (x_j = y_j) \wedge (x_i = x_k) \wedge \neg(y_i = y_k) \wedge (y_{i+1} = y_k) \Big) \end{aligned}$$

Then $I(\mathbf{A}_n)$ is the tree-blowing of F (in [7] this operation on rooted trees is called "branching"). By Corollary 5.1 these graphs form a strongly polynomial sequence.

5.1.7 Union of stars of orders $1, \dots, P(n)$

Let $I = (2, \iota, \rho)$ be the graphical interpretation scheme defined by

$$\begin{aligned} \iota(x,y) :& S_1(x,y) \vee (x = y) \\ \rho(x_1, y_1, x_2, y_2) :& (y_1 = y_2) \wedge [(x_1 = y_1) \wedge S_1(x_2, y_2) \vee (x_2 = y_2) \wedge S_1(x_1, y_1)] \end{aligned}$$

Then, for $\mathbf{A}_n = \mathbf{T}_{P(n)}$, we have

$$I(\mathbf{A_n}) = \bigcup_{i=1}^{P(n)} \mathbf{S}_i,$$

where $\mathbf{S}_i$ is the star of order i. By Corollary 5.1 the sequence $(\bigcup_{i=1}^{P(n)} \mathbf{S}_i)_{n\in\mathbb{N}}$ is strongly polynomial.

5.1.8 Half graphs

Let $\mathbf{A}_n = \mathbf{E} \oplus \mathbf{E} \oplus \mathbf{T}_n$. Consider the graphical interpretation scheme $I = (2, \iota, \rho)$ where:

$$\begin{aligned} \iota(x_1, x_2) : \quad & U_1^T(x_1) \wedge \neg U_1^T(x_2) \\ \rho(x_1, x_2, y_1, y_2) : & [S_1(x_1, y_1) \vee (x_1 = y_1)] \wedge U_1^E(x_2) \wedge U_2^E(y_2) \\ & \vee \; [S_1(y_1, x_1) \vee (x_1 = y_1)] \wedge U_1^E(y_2) \wedge U_2^E(x_2) \end{aligned}$$

The graph $I(\mathbf{A}_n)$ is the half graph on $2n$ vertices (see Fig. 1).

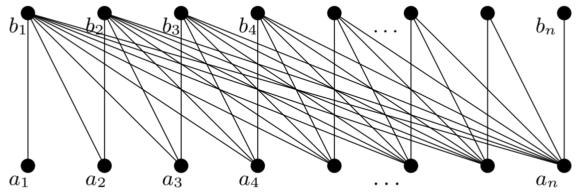
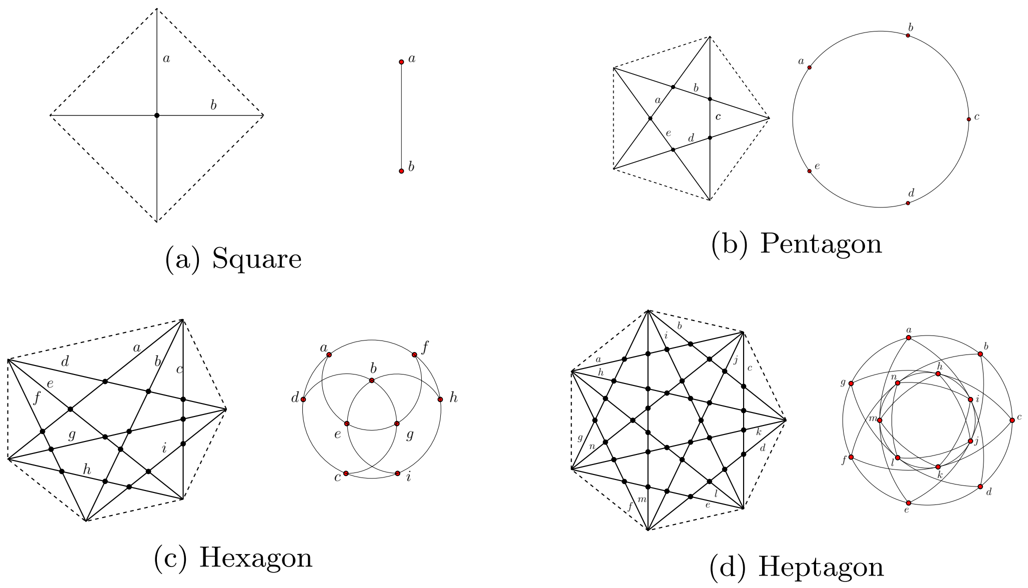


Figure 1: Half graphs form a strongly polynomial sequence

By Corollary 5.1 the sequence of half graphs on $2n$ vertices is strongly polynomial. This example demonstrates that a strongly sequence of graphs (G_n) need not have the property that the number of automorphisms of G_n grows with n.

5.1.9 Intersection graphs of chords



(a) Square

(b) Pentagon

(c) Hexagon

(d) Heptagon

Figure 2: Intersection graphs of chords of a convex n-gon form a strongly polynomial sequence

Let $\mathbf{A}_n = \mathbf{T}_n$. Consider the graphical interpretation scheme $I = (2, \iota, \rho)$ where:

$$\begin{aligned} \iota(x_1, x_2) : \quad & S_1(x_1, x_2) \\ \rho(x_1, x_2, y_1, y_2) : & S_1(x_1, y_1) \wedge S_1(y_1, x_2) \wedge S_1(x_2, y_2) \end{aligned}$$

The graph $I(\mathbf{A}_n)$ is the intersection graph of chords of a convex n-gon (see Figure 2) and by Corollary 5.1 these graphs form a strongly polynomial sequence.

5.1.10 Fractional cliques

Recall that the *circular chromatic number* $\chi_c(G)$ of a graph G is the minimum over all rational numbers $\frac{n}{k}$ such that there exists a map from $V(G)$ to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with the property that adjacent vertices map to elements at distance at least k apart. Alternatively, the circular chromatic number can be defined as the minimum ratio $\frac{p}{q}$ such that G is homomorphic to $K_{p/q}$, where $K_{p/q}$ denotes the graph with vertex set $\{0, \ldots, p-1\}$, where two vertices x, y are adjacent if $q \leq |x-y| \leq p-q$ (see Fig. 3). For $p \geq 2q$, a homomorphism from G to $K_{p/q}$ is called a (p,q)-*colouring* of G.

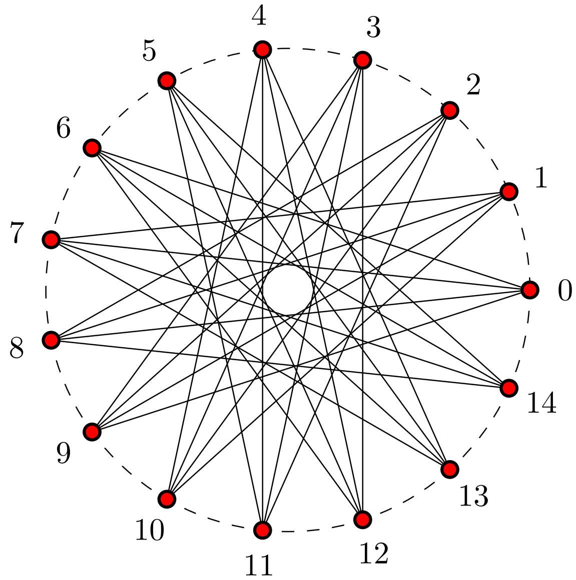


Figure 3: The graph $K_{15/6}$ has vertex set $\{0, \ldots, 14\}$ and edges ij precisely when $6 \leq |i-j| \leq 9$

Proposition 5.3. *For every $p \geq 2q$, the sequence $(K_{pn/qn})_{n \in \mathbb{N}}$ is strongly polynomial.*

In other words, for every graph G and every integer $p \geq 2q$, the number of (pn, qn)-colourings of G is polynomial in n.

Proof. Let $\mathbf{A_n} = \mathbf{T}_p \oplus \mathbf{T}_n$. Elements of $K_{pn/qn}$ will be the elements (a, b) of A_n^2 such that $a \in \mathbf{T}_p$ and $b \in \mathbf{T}_n$. A vertex (a, b) will be adjacent to a vertex (a', b') if

- $a' = (a+q) \bmod p$ and $b' \geq b$,
- or a' between $(a+q+1) \bmod p$ and $(a+p-q-1) \bmod p$,
- or $a' = (a-q) \bmod p$ and $b' \leq b$.

The result now follows from Corollary 5.1. $\square$

5.1.11 Carousel tournaments

The *carousel tournament* R_{2n+1} is the unique (up to isomorphism) tournament of order $2n+1$ that is both *balanced* (each vertex has the same indegree as out-degree) and *locally transitive* (the in-neighbourhood and the out-neighbourhood of each vertex are both transitive), see Figure 4.

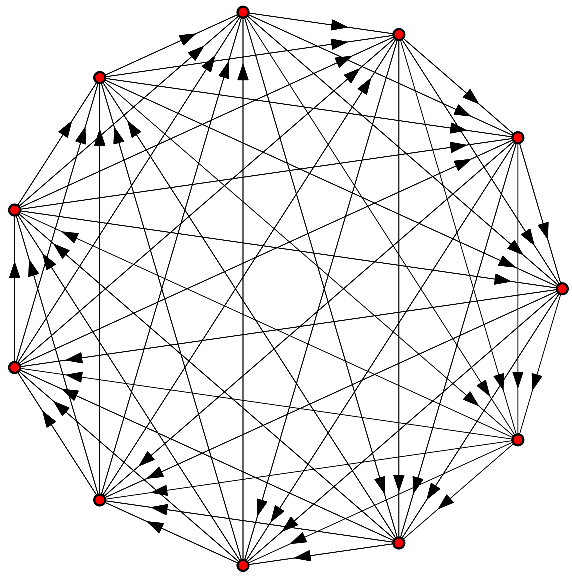


Figure 4: The carousel tournament R_{11}

Proposition 5.4. *The sequence $(R_{2n+1})_{n\in\mathbb{N}}$ is a strongly polynomial sequence of digraphs.*

Proof. Let $\mathbf{A_n} = \mathbf{E}\oplus\mathbf{E}\oplus\mathbf{T}_n$. Let α, β the called special vertices and let $1, \ldots, n$ be the elements of the tournament. The vertex set of the carousel R_{2n+1} is the subset of A_n^2 formed by pairs of the form (α, i), (β, i), or (α, α). Arcs are pairs $((a, b), (a', b'))$ such that:

- either $a = a'$, $b, b' \in \{1, \ldots, n\}$ and $b' \geq b$,
- or $a = \alpha, a' = \beta$, $b, b' \in \{1, \ldots, n\}$ and $b' \leq b$,
- or $a = \beta$, $b \in \{1, \ldots, n\}$, and $a' = b' = \alpha$,
- or $a = b = a' = \alpha$ and $b' \in \{1, \ldots, n\}$.

An appeal to Corollary 5.1 completes the proof. □

Remark 5.5. Proposition 5.4 enables an easy proof of [6, Prop. 4.1], that the sequence (R_{2n+1}) is L-convergent, the limiting homomorphism density of a digraph D in (R_{2n+1}) being given by the coefficient of $n^{|D|}$ in the polynomial $\hom(D, R_{2n+1})$.

5.2 Sequences of bounded degree graphs

In this section we completely characterize strongly polynomial sequences of graphs of uniformly bounded degree.

Theorem 5.6. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of graphs of uniformly bounded degree. Then the following conditions are equivalent:*

1. *the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is strongly polynomial;*
2. *there is a finite set $\{\mathbf{F}_1, \dots, \mathbf{F}_k\}$ of graphs and polynomials $P_1, \dots, P_k$ such that*
$$\mathbf{A}_n = \sum_{i=1}^{k} \bigsqcup^{P_i(n)} \mathbf{F}_i;$$
3. *the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a QF-interpretation of a basic sequence.*

Proof. Assume that the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is strongly polynomial. Let $P(n) = |A_n|$, $d = \deg P$ and, for a connected graph $\mathbf{F}$, let $P_{\mathbf{F}}(n) = \mathrm{ind}(\mathbf{F}, \mathbf{A}_n)$ be the number of copies of $\mathbf{F}$ that are induced subgraphs in $\mathbf{A}_n$, which is a polynomial in n by Theorem 2.5(iv). As $\Delta(\mathbf{A}_n) \le D$ for some fixed bound D, we have $P_{\mathbf{F}}(n) \le D^{|F|-1} P(n)$ and so all the polynomials $P_{\mathbf{F}}$ have degree at most d. It follows that $P_{\mathbf{F}} \neq 0$ if and only if there exists $i \le d+1$ such that $P_{\mathbf{F}}(i) \neq 0$, that is, if and only if $\mathbf{F}$ is an induced subgraph of $\bigcup_{i=1}^{d+1} \mathbf{A}_i$. Let $\mathbf{F}_1, \dots, \mathbf{F}_k$ be the connected induced subgraphs of $\bigcup_{i=1}^{d+1} \mathbf{A}_i$. As every connected component of $\mathbf{A}_n$ belongs to $\{\mathbf{F}_1, \dots, \mathbf{F}_k\}$, we infer that there exist polynomials $P_1, \dots, P_k$ such that $\mathbf{A}_n = \sum_{i=1}^{k} \bigsqcup^{P_i(n)} \mathbf{F}_i$, as can be proved by induction on k as follows. Let $\mathbf{F}$ be a maximal connected induced subgraph of $\bigcup_{i=1}^{d+1} \mathbf{A}_i$. Without loss of generality, we can assume $\mathbf{F} = \mathbf{F}_k$. Let $P_k = P_{\mathbf{F}_k}$. Then $\mathbf{A}_n$ contains $P_k(n)$ disjoint copies of $\mathbf{F}_k$, each of them being a connected component of $\mathbf{A}_n$ (by maximality of $\mathbf{F}_k$). Hence we can define the sequence $(\mathbf{B}_n)_{n\in\mathbb{N}}$ by requiring that $\mathbf{A}_n = \mathbf{B}_n + \bigsqcup^{P_k(n)} \mathbf{F}_k$. The sequence $(\mathbf{B}_n)_{n\in\mathbb{N}}$ is obviously strongly polynomial. Moreover, the connected induced subgraphs of $\bigcup_{i=1}^{d+1} \mathbf{B}_i$ form a proper subset of the set of connected induced subgraphs of $\bigcup_{i=1}^{d+1} \mathbf{A}_i$. Without loss of generality, these induced subgraphs are $\mathbf{F}_1, \dots, \mathbf{F}_\ell$ (for some $\ell < k$). Hence, by induction hypothesis, there are polynomials $P_1, \dots, P_\ell$ such that $\mathbf{B}_n = \sum_{i=1}^{\ell} \bigsqcup^{P_i(n)} \mathbf{F}_i$. Thus $\mathbf{A}_n = \sum_{i=1}^{\ell} \bigsqcup^{P_i(n)} \mathbf{F}_i + \bigsqcup^{P_k(n)} \mathbf{F}_k$.

Assume that there is a finite set $\{\mathbf{F}_1, \dots, \mathbf{F}_k\}$ of graphs and polynomials $P_1, \dots, P_k$ such that $\mathbf{A}_n = \sum_{i=1}^{k} \bigsqcup^{P_i(n)} \mathbf{F}_i$. Then the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is obviously a QF-interpretation of a basic sequence.

Assume that the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a QF-interpretation of a basic sequence. Then by Corollary 5.1 it is strongly polynomial, . ☐

Remark 5.7. In the case of sequences of graphs with uniformly bounded degree, one can consider even simpler basic sequences: A *simplest basic structure* is a structure $\mathbf{B} = \oplus_{i=1}^{k} \mathbf{E}_{N_i}$, where $\mathbf{E}_j$ is the graph with j vertices and no edges. A *simplest basic sequence* is a sequence $(\mathbf{B}_n)_{n\in\mathbb{N}}$ of simplest basic structures $\mathbf{B}_n = \oplus_{i=1}^{k} \mathbf{E}_{Q_i(n)}$, for some polynomials Q_i ($i = 1, \dots, k$). Then it follows from Theorem 5.6 that a sequence of graphs of uniformly bounded degree is strongly polynomial if and only if it is a QF-interpretation of a simplest basic sequence.

6 Left limits of strongly polynomial sequences

Lovász and Szegedy [13] define a graph property (or equivalently a class of graphs) $\mathcal{C}$ to be *random-free* if every left limit of graphs in $\mathcal{C}$ is random-free.

They prove the following:

Theorem 6.1 (Lovász and Szegedy [13]). *A hereditary class $\mathcal{C}$ is random-free if and only if there exists a bipartite graph F with bipartition (V_1, V_2) such that no graph obtained from F by adding edges within V_1 or within V_2 is in $\mathcal{C}$.*

This theorem has, in our setting, the following corollary, which gives a necessary condition for a sequence of graphs to be strongly polynomial.

Theorem 6.2. *Every strongly polynomial sequence of graphs $(G_n)_{n\in\mathbb{N}}$ converges to a random-free graphon.*

Proof. Consider a strongly polynomial sequence of graphs $(G_n)_{n\in\mathbb{N}}$. Let $P(n) = |G_n|$ and $d = \deg P$. For every graph F, the probability that a random map from F to G_n is a homomorphism is a fixed rational function of n, hence converges as $n \to \infty$. It follows that the sequence $(G_n)_{n\in\mathbb{N}}$ converges to some graphon W.

For $k \in \mathbb{N}$, consider the bipartite graph $F_k = (V_1, V_2, E)$, where $|V_1| = k$, $|V_2| = 2^k$, and the neighbourhoods of vertices in V_2 are pairwise distinct. Let F' be any graph obtained from F_k by adding some edges whose endpoints both belong to V_1 or both to V_2. There are $\binom{2^k}{k}$ ways to choose k vertices from V_2, which together with the k vertices of V_1 induce a subgraph of F' of order $2k$, which is unique up to the choice of the ordered part of k vertices corresponding to V_1. Hence there are at least $\binom{2^k}{k}/(k!\binom{2k}{k}) = 2^{k^2(1-o(1))}$ distinct induced subgraphs of F' of order $2k$. Thus, if a hereditary class $\mathcal{C}$ of graphs is not random-free, there exists for every integer k, according to Theorem 6.1, a graph F' derived from F_k that belongs to $\mathcal{C}$. Hence the number of graphs of order $2k$ in $\mathcal{C}$ is at least $2^{k^2(1-o(1))}$.

To the sequence (G_n) corresponds a hereditary class $\mathcal{F} = \{F : \exists n\ F \subseteq_i G_n\}$, consisting of graphs F that occur as an induced subgraph of some G_n. If a graph F of order k belongs to $\mathcal{F}$, then it is an induced subgraph of a graph G_n with $n \le kd + 1$. Indeed, the degree of the polynomial P_F counting F is at most kd, hence if $P_F(n) = 0$ for every $n \le kd + 1$, then $P_F = 0$. It follows that the number of induced subgraphs of order k is bounded by $\sum_{i=1}^{kd+1} \binom{P(i)}{k} = 2^{o(k^2)}$. It follows that every strongly polynomial sequence converges to a random-free graphon. □

Thus strongly polynomial sequences are special converging sequences. They are very special sequences as most sequences converging not only fail to be strongly polynomial, but also fail to converge to the limit of a strongly polynomial sequence. In fact the set of graphons that appear as limits of strongly polynomial sequences is at most countable (up to equivalence).

7 Going further

We have seen that QF-interpretations of basic sequences form strongly polynomial sequences. We now extend this construction to the generalized basic sequences of Definition 7.3 below, as a way to generate new strongly polynomial sequences from old.

Definition 7.1. Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of λ-structures. Let λ^+ be the signature obtained from λ by adding a new binary relation symbol S.

For $n \in \mathbb{N}$, the λ^+-structure $\mathbf{T}\langle\mathbf{A}\rangle_n$ is obtained from the disjoint union $\sum_{i=1}^n \mathbf{A}_i$ by defining a binary relation S as follows:

- for all vertices $x \in A_i$ and $y \in A_j$, $\mathbf{T}\langle\mathbf{A}\rangle_n \models S(x,y)$ if $i < j$.

For example, for the constant sequence (K_1) of single point structures we have $\mathbf{T}\langle K_1\rangle_n = \vec{T}_n$.

Lemma 7.2. *Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a strongly polynomial sequence of λ-structures. Let λ^+ be the signature obtained from λ by adding a new binary relation symbol S.*

Then $(\mathbf{T}\langle\mathbf{A}\rangle_n)_{n\in\mathbb{N}}$ is a strongly polynomial sequence of λ^+-structures.

Proof. Let $\mathbf{F}$ be a connected λ^+-structure. Say that $\mathbf{F}$ is nice if F can be partitioned as $F = \bigcup_{i=1}^k F_i$, with the property that for every $(x,y) \in F_i \times F_j$ it holds that $\mathbf{F} \models S(x,y)$ if and only if $i < j$. Then the number $\mathrm{inj}(\mathbf{F}, \mathbf{T}\langle\mathbf{A}\rangle_n)$ of injective homomorphisms $f : \mathbf{F} \to \mathbf{T}\langle\mathbf{A}\rangle_n$ is given by

$$\mathrm{inj}(\mathbf{F}, \mathbf{T}\langle\mathbf{A}\rangle_n) = \begin{cases} 0, & \text{if } \mathbf{F} \text{ is not nice,} \\ \displaystyle\sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^k \mathrm{inj}(\mathbf{F}_j, \mathbf{A}_{i_j}), & \text{otherwise,} \end{cases}$$

where $\mathbf{F}_j$ is the substructure of $\mathbf{F}$ induced on F_j. As $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is strongly polynomial, there are polynomials $P_1, \ldots, P_k$ such that

$$\mathrm{inj}(\mathbf{F}_j, \mathbf{A}_n) = P_j(n).$$

For every $n \in \mathbb{N}$, we have

$$\sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^k P_j(i_j) = \sum_{i_1=1}^n P_1(i_1) \left(\sum_{i_2=i_1+1}^n P_2(i_2) \left(\ldots \sum_{i_k=i_{k-1}+1}^n P_k(i_k) \ldots \right)\right).$$

But for each k there exists a polynomial Q_k such that $\sum_{i_k=i_{k-1}+1}^n P_k(i_k) = Q_k(n) - Q_k(i_{k-1})$, in which $i_0 = 0$. By induction on k, it follows that there exists a polynomial $Q_\mathbf{F}$ such that for every $n \in \mathbb{N}$ we have

$$Q_\mathbf{F}(n) = \begin{cases} 0, & \text{if } \mathbf{F} \text{ is not nice,} \\ \displaystyle\sum_{1\le i_1<\cdots<i_k\le n} \prod_{j=1}^k P_j(i_j), & \text{otherwise.} \end{cases}$$

It follows that the sequence $(\mathbf{T}\langle\mathbf{A}\rangle_n)_{n\in\mathbb{N}}$ is strongly polynomial. □

Definition 7.3. A *generalized basic structure with parameter*

$$(((\mathbf{A}_n^1)_{n\in\mathbb{N}}, \ldots, (\mathbf{A}_n^k)_{n\in\mathbb{N}}), (\mathbf{B}^1, \ldots, \mathbf{B}^\ell))$$

is any structure of the form

$$\mathbf{C} = \bigoplus_{i=1}^{\ell} \mathbf{B}^i \oplus \bigoplus_{j=1}^{k} \mathbf{T}\langle\mathbf{A}^j\rangle_{N_j},$$

with $N_1, \dots, N_k \in \mathbb{N}$. These integers will be also be denoted by $N_1(\mathbf{C}), \dots, N_k(\mathbf{C})$.

A *generalized basic sequence* is a sequence $(\mathbf{C}_n)_{n\in\mathbb{N}}$ of generalized basic structures $\mathbf{C}_n$ with the same parameter $(((\mathbf{A}_n^1)_{n\in\mathbb{N}}, \dots, (\mathbf{A}_n^k)_{n\in\mathbb{N}}), (\mathbf{B}^1, \dots, \mathbf{B}^\ell))$, such that there are non-constant polynomials Q_i, $1 \le i \le k$ with $Q_i(n) = N_i(\mathbf{C}_n)$ (for every $1 \le i \le k$ and $n \in \mathbb{N}$).

Theorem 7.4. *For every generalized basic sequence $(\mathbf{C}_n)_{n\in\mathbb{N}}$ and every QF-interpretation scheme I, the sequence $(I(\mathbf{C}_n))_{n\in\mathbb{N}}$ is strongly polynomial.*

Proof. As the sequence $(\mathbf{T}\langle\mathbf{A}\rangle_n)_{n\in\mathbb{N}}$ is strongly polynomial, this theorem is a direct consequence of Lemmas 2.13 and 2.11. □

8 Concluding remarks

An as yet unresolved problem arising from this paper is to establish whether the strongly polynomial sequences we have constructed from basic sequences constitute the general case, as suggested by Theorem 5.6.

A strongly polynomial sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *induced monotone* if G_n is an induced subgraph of G_{n+1} for each $n \in \mathbb{N}$. Equivalently, $(G_n)_{n\in\mathbb{N}}$ is induced monotone if there exists a countable graph G such that G_n is the subgraph induced by vertices $1, \dots, f(n)$, where f is a monotone (non-decreasing) function.

Problem 8.1. *Can all strongly polynomial induced monotone sequences of graphs be obtained by QF-interpretation schemes from generalized basic sequences (as defined in Definition 7.3)?*

There is more to this than meets the eye. By Theorem 6.2, we know that every strongly polynomial sequence converges to a random-free graphon. In some sense, we are asking here whether, under the stronger assumption that the sequence of graphs has an inductive countable limit, the countable limit itself may be an interpretation of a countable "basic" structure.

The polynomial graph invariants defined by strongly polynomial sequences that have received most attention – such as the chromatic polynomial, Tutte polynomial and independence polynomial – satisfy reduction formulas, i.e., size-reducing recurrences such as edge deletion-contraction and vertex-neighbourhood deletion. The existence of such a recurrence enables the recursive computation of the invariant on G by applying local operations to G and is often correlated to yielding rich combinatorial information and to having interpretations across different combinatorial fields (such as the Tutte polynomial in knot theory and statistical physics). The tantalizing question remains, therefore: which other polynomial graph invariants defined by QF-interpretation of basic sequences have this property? Are there undiscovered graph polynomials among the large class we define in this paper which have a reduction formula and might be as fruitfully studied as the chromatic polynomial?

Acknowledgements

The authors would like to thank the referees for their careful reading of the manuscript and for their constructive comments. Remark 5.7 is due to Benny Godlin.

References

[1] D. J. Aldous. Exchangeability and related topics. *École d'été de probabilités de Saint-Flour, XIII1983*, vol. 1117 of Lecture Notes in Math., pp. 1–198. Springer, Berlin, 1985

[2] I. Averbouch, B. Godlin and J.A. Makowsky, A most general edge elimination polynomial, in: H. Broersma, T. Erlebach, T. Friedetzky, D. Paulusma (eds.), *Graph-Theoretic Concepts in Computer Science*, 34th International Workshop, WG2008, Durham, UK, June/July 2008, Lect. Notes Comput. Sci. **5344** (2008), 31–42

[3] L. Babai, Graph Isomorphism in Quasipolynomial Time, arXiv:1512.03547 [cs.DS] (11 Dec., 2015)

[4] C. Borgs, J. Chayes, L. Lovász, V.T. Sós and K. Vesztergombi, Counting graph homomorphisms. In: M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas, P. Valtr (eds.), *Topics in Discrete Mathematics*, Springer-Verlag, Berlin, 2006, 315–371

[5] A. K. Chandra and P. M. Merlin, Optimal implementation of conjunctive queries in relational data bases. *Proceedings of the Ninth Annual ACM Symposium on the Theory of Computing (STOC '77)*, ACM, New York, 1977, 77–90

[6] L. N. Coregliano, Quasi-carousel tournaments, arXiv:1503.04124 [math.CO] (15 Apr., 2015)

[7] D. Garijo, A. Goodall and J. Nešetřil, Polynomial graph invariants from graph homomorphisms, *Discrete Math.* To appear.

[8] B. Grünbaum, Acyclic colorings of planar graphs, *Israel J. Math.* **14** (1973), 390–408

[9] P. de la Harpe and F. Jaeger, Chromatic invariants for finite graphs: theme and polynomial variations, *Lin. Algebra Appl.* **226–228** (1995), 687–722

[10] W. Hodges, *A Shorter Model Theory*, Cambridge Univ. Press, Cambridge, 1997

[11] T. Kotek, J.A. Makowsky, B. Zilber, On counting generalized colorings. In: M. Grohe, J.A. Makowsky (eds.), *Model Theoretic Methods in Finite Combinatorics*, Contemporary Mathematics, vol. 558, AMS, 2011, pp. 207–242

[12] D. Lascar, *La théorie des modèles en peu de maux*, Nouvelle bibliothèque mathématique **10**, Cassini, 2009

[13] L. Lovász and B. Szegedy, Regularity partitions and the topology of graphons. *An Irregular Mind, Szemerédi is 70*, János Bolyai Math. Soc. Stud., 21, Budapest, 2010, 415–446

[14] J.A. Makowsky, From a zoo to a zoology: towards a general theory of graph polynomials, *Theory Comput. Syst.* **43** (2008), 542–562

[15] J.A. Makowsky and B. Zilber, Polynomial invariants of graphs and totally categorical theories, MODNET Preprint No. 21, 2006

[16] P. Tittmann, I. Averbouch and J.A. Makowsky, The enumeration of vertex-induced subgraphs with respect to the number of components, *European J. Combin.* **32**:7 (2011), 954–974