TOWARDS A CHARACTERIZATION OF UNIVERSAL CATEGORIES

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ABSTRACT. In this note we characterize, within the framework of the theory of finite set, those categories of graphs that are *algebraic universal* in the sense that every concrete category fully embeds in them. The proof of the characterization is based on the sparse–dense dichotomy and its model theoretic equivalent.

1. INTRODUCTION

A category $\mathcal{K}$ is *algebraic universal* if every concrete category embeds in it. (In this paper, embeddings are understood as full and faithful functors) The name comes from examples: algebraic universal categories include simple algebraic structures as well as the class of all graphs (sets with one binary relation). Algebraic universal categories have been the subject of intensive studies [1, 26]. Particularly, many subcategories of the category of graphs were shown to be algebraic universal, too [2, 5, 13, 26].

The aim of this note is to provide a characterization of those subcategories of the category of graphs that are universal. Unexpectedly this is related to (and in fact coincides with) the characterization of somewhere dense classes of graphs. All these notions will be introduced in Section 2.

At this place let us remark that we deal only with finite graphs and categories induced by them, so this paper is in fact written in the theory of finite sets (so $\mathbb{N}$ is a proper class here), see Section 4.

The main result of this note is the following: Denote by **Gra** the category of all finite undirected graphs, and by $\overrightarrow{\textbf{Gra}}$ the category of all finite oriented graphs. Recall that an *oriented graph* is a directed graph in which at most one arc exists between any two vertices.

Theorem 1. *For a monotone subcategory $\mathcal{K}$ of **Gra** the following three statements are equivalent:*

> (1) *There exists a subcategory $\overline{\mathcal{K}}$ of $\overrightarrow{\textbf{Gra}}$, each member of which is an orientation of a member of $\mathcal{K}$, which embeds the category $\overrightarrow{\textbf{Gra}}$.*
> (2) *There exists a subcategory $\overline{\mathcal{K}}$ of $\overrightarrow{\textbf{Gra}}$, each member of which is an orientation of a member of $\mathcal{K}$, which embeds the simplicial category Δ.*
> (3) *$\mathcal{K}$ is somewhere dense.*

In other words, some class of orientations of graphs from $\mathcal{K}$ is universal (in the theory of finite sets) if and only if the class $\mathcal{K}$ is somewhere dense.

This in turn leads to a new, high level, algebraic characterization of somewhere dense classes. Yet another one in the already long list, see [24, 25].

The paper is organised as follows: In Section 2 we recall all the relevant notions and put the universality question in the context of category theory related to concreteness and representation of posets. In Section 3 we prove the main result by a combination of model theory and combinatorial methods. In Section 4 we recast the problem of universality in the context of the classification of sparse classes of graphs, and display a perhaps surprising gap in the descriptive complexity of classes representing groups, monoids, and categories.

2. Preliminaries

First we recall several notions of category theory.

A category is *concrete* if it is isomorphic to a subcategory of the category **Set** of sets and mappings. A necessary condition for a category to be concrete is *Isbell condition* [17]. This condition was proved to be sufficient by Freyd [10] and, through the explicit construction of a faithful functor to **Set**, by Vinárek [29]. Vinárek's construction has, moreover, the following property: for countable categories with finite sets of morphisms between fixed objects, the functor has finite values. Thus Freyds' theorem holds also in the theory of finite sets. Precisely, if the considered class is countable and the set of homomorphisms between any two objects is finite, then the class is isomorphic to a subcategory of the category of finite sets if and only if Isbell's condition holds. See [14] for a concise description of these results.

In this context, another interesting result is Kučera's theorem [18], which asserts that every category is a factorization of a concrete one (like classes of homotopy equivalent maps, which was the original motivation of [10]). Also this theorem holds in its version for the theory of finite sets.

As a culmination of researches by Prague category group in the sixties, it has been proved that the category **Gra** of all graphs (finite or infinite) with homomorphisms between them is algebraic universal for all concrete categories. Explicitly, for every concrete category $\mathcal{K}$ there is an embedding of $\mathcal{K}$ into **Gra**.

These results led to an intensive research, and various subcategories of **Gra** were shown to be algebraic universal.

The basic techniques used in these proofs was model-theoretical first-order interpretation, then called šíp, indicator, or replacement construction.
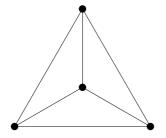
It is perhaps surprising that in this paper we can provide a characterization of monotone subcategories of **Gra** that are algebraic universal. Here *monotone* means that the class (of graphs) is closed under taking (non necessarily induced) subgraphs.

As this paper deals with finite models we restrict from now in the setting of theory of finite sets, thus to finite graphs and to embedding into the category of finite graphs.

In order to formulate our main results we have to recall the basics of the nowhere dense–somewhere dense dichotomy. For a comprehensive treatment, see e.g. [24] or [25].

Somewhere dense classes of graphs were introduced by the authors in [22, 23]. Recall that a *directed graph* is a relational structure with a single binary (non-necessarily symmetric) adjacency relation, whose elements are called *arcs*. The domain of a directed graph is its *vertex set*. An *undirected graph* (resp. an *oriented graph*) is a directed graph such that the adjacency relation is symmetric (resp. anti-symmetric). In an undirected graph, we speak about *edges* rather than arcs, the set $\{u, v\}$ being an edge if the vertices u and v are adjacent. When not specified, graphs are assumed to be undirected.

For $n \in \mathbb{N}$ the *complete graph* K_n is the graph with n vertices, where every two distinct vertices are adjacent. A path of length n is a graph with n vertices $v_1, \ldots, v_n$ such that v_i is adjacent to v_{i+1} (and to no v_j for $j > i + 1$) for each $1 \leq i < n$. For a graph G and an integer p, the *p-th subdivision* of G, denoted $\mathrm{Sub}_p(G)$ is the graph obtained from G by replacing each edge by a path of length $p + 1$ (see Fig. 1).
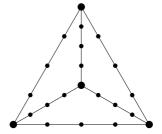

FIGURE 1. The complete graph K_4 (left) and its 3-subdivision $\mathrm{Sub}_3(K_4)$ (right).

A class of graphs $\mathcal{C}$ is *nowhere dense* if, for every integer p there exists an integer $N(p)$ such that the p-th subdivision $\mathrm{Sub}_p(K_{N(p)})$ of $K_{N(p)}$ is a subgraph of no graph in $\mathcal{C}$, and the class $\mathcal{C}$ is *somewhere dense*, otherwise. So, a monotone class $\mathcal{C}$ is nowhere dense if an only if there exists $N : \mathbb{N} \to \mathbb{N}$ such that for every $p \in \mathbb{N}$ it holds $\mathrm{Sub}_p(K_{N(p)}) \notin \mathcal{C}$. Nowhere dense classes found various applications in designing fast (almost linear) algorithms [21, 12, 8]. Particular

cases of nowhere dense classes are *classes with bounded expansion* [20]. These are characterized by the property that for every integer p there exists an integer $N(p)$ such that no p-th subdivision of a graph with minimum degree at least $N(p)$ is a subgraph of a graph in $\mathcal{C}$. Such classes have strong structural and algorithmic properties [24].

Let $\mathcal{C}$ be a class of structures of a fixed signature. A first-order formula $\phi(\mathbf{x}, \mathbf{y})$ is said to have the *order property* with respect to $\mathcal{C}$ if it has the *n-order property* for all n, i.e. if for every n there exist a structure $\mathbf{M} \in \mathcal{C}$ and tuples $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}, b_0, \ldots, \mathbf{b}_{n-1}$ of elements of $\mathbf{M}$ such that $\mathbf{M} \models \phi(\mathbf{a}_i, \mathbf{b}_j)$ holds if and only if $i < j$. A class $\mathcal{C}$ of structures is called *stable* if there is no such formula with respect to $\mathcal{C}$. It is easy to see that $\mathcal{C}$ is stable if and only if there is no formula $\psi(\mathbf{u}, \mathbf{v})$ with $|\mathbf{u}| = |\mathbf{v}|$, such that for every n there exist a structure $\mathbf{M} \in \mathcal{C}$ and tuples $\mathbf{c}_0, \ldots, \mathbf{c}_{n-1}$ of elements of $\mathbf{M}$ such that $\mathbf{M} \models \psi(\mathbf{c}_i, \mathbf{c}_j)$ holds if and only if $i < j$, i.e. ψ orders the tuples linearly. Stability and the (n-)order property come from stability theory [28, 9], where they are defined for the class of models of a complete first-order theory. In [3], Adler and Adler prove the following theorem (see also [25]) (which we state here in the particular case we are interested in).

Theorem 2. *Let $\mathcal{C}$ be a monotone class of directed graphs, and let $\underline{\mathcal{C}}$ be the class of the underlying undirected graphs. The following conditions are equivalent.*

  (1) $\underline{\mathcal{C}}$ is nowhere dense;
  (2) $\mathcal{C}$ is stable;

This interplay of model theoretic and combinatorial notions is the key to our main result.

3. CHARACTERIZATION

In view of the context of our main result (outlined in Section 1) it suffices to prove the following two results.

Lemma 3. *Let $\mathcal{D}$ be a monotone somewhere dense class of undirected graphs. Then there exists a class $\mathcal{C}$ of oriented graphs, each member of which is an orientation of a graph in $\mathcal{D}$, which embeds the category $\overrightarrow{\mathbf{Gra}}$ of oriented graphs.*

Proof. Let d be such that $\mathcal{C}$ contains the d-subdivision of every complete graph K_n (here we use the assumption that $\mathcal{C}$ is monotone). Let (I, a, b) be the circuit of length $3(d + 1)$, where vertices a, b are linked by a directed path (from a to b) of length $d + 1$. For a given oriented graph $\vec{G}$, denote by $\vec{G} * (I, a, b)$ the directed graph which arises from $\vec{G}$ by replacing every arc (u, v) of $\vec{G}$ by a copy of (I, a, b) in such a way that a is identified to u and b to v (all other vertices in distinct copies being distinct). It is easy to check that the underlying undirected graph of $\vec{G} * (I, a, b)$ belongs to $\mathcal{C}$.

The only circuits of $G * (I, a, b)$ with length at most $3(d+1)$ occur as copies of (I, a, b). It follows that any homomorphism $f : \vec{G} * (I, a, b) \to \vec{H} * (I, ab)$ is induced in a unique way by a homomorphism $g : \vec{G} \to \vec{H}$. $\qquad\square$

Lemma 4. *If a class of directed graphs embeds the simplicial category Δ, then it is somewhere dense.*

Proof. Let $\mathcal{C}$ be a class of directed graphs that represents the the category Δ. For sake of simplicity, we assume that every directed graph $\vec{G}$ we consider has vertex set $0, 1, 2, \ldots, |\vec{G}| - 1$, and we denote by $E(x, y)$ the relation expressing the existence of an arc from x to y.

Then there is a functor Φ, mapping each ordinal $[n] = \{0, 1, \ldots, n\}$ to a directed graph $\Phi([n]) \in \mathcal{C}$, and bijectively mapping order preserving maps $f : [i] \to [j]$ into homomorphisms $\Phi(f) : \Phi([i]) \to \Phi([j])$ in such a way that $\Phi(f \circ g) = \Phi(f) \circ \Phi(g)$.

Let $\vec{G}_n = \Phi([n])$, let $a = |\vec{G}_0| - 1$ and $b = |\vec{G}_1| - 1$. We define the formula

$$\nu(x_0, \ldots, x_a) := \bigwedge_{\vec{G}_0 \models E(i,j)} E(x_i, x_j),$$

which asserts that $i \mapsto x_i$ is a homomorphism from $\vec{G}_0$.

There are exactly two order preserving maps from $[0]$ to $[1]$, namely $f_s : 0 \mapsto 0$, and $f_t : 0 \mapsto 1$. Let $\phi_s = \Phi(f_s)$ and $\phi_t = \Phi(f_t)$.

Then we define

$$\eta(x_0, \ldots, x_a, y_0, \ldots, y_a) :=$$

$$(\exists z_0 \ldots z_b) \quad \left[\bigwedge_{i=0}^{a}(x_i = z_{\phi_s(i)}) \wedge \bigwedge_{i=0}^{a}(y_i = z_{\phi_t(i)}) \wedge \bigwedge_{\vec{G}_1 \models E(i,j)} E(z_i, z_j) \right]$$

The meaning of formula η is as follows: $\vec{G} \models \eta(x_0, \ldots, x_a, y_0, \ldots, y_a)$ expresses that there exist a homomorphism $h : \vec{G}_1 \to \vec{G}$ and homomorphisms $g_s, g_t : \vec{G}_0 \to \vec{G}$, such that $g_s(i) = x_i$, $g_t(i) = y_i$, $g_s = h \circ \phi_s$, and $g_t = h \circ \phi_t$. In other words, naming $z_i = h(i)$, there exist $z_0, \ldots, z_b$ such that $i \mapsto z_i$ is a homomorphism $\vec{G}_1 \to \vec{G}$, $x_i = z_{\phi_s(i)}$, and $y_i = z_{\phi_t(i)}$. (Note that ϕ_s and ϕ_t are known to be homomorphisms.)

Let n be an ordinal. There are exactly $n+1$ order preserving maps $g_i : [0] \to [n]$, that are naturally ordered in such a way that for every $i, j \in [n]$ it holds $i < j$ if and only if $g_i(0) < g_j(0)$. In other words, for every two order preserving maps $g_i, g_j : [0] \to [n]$ there exists an order preserving map $h : [1] \to [n]$ such that $g_i = h \circ f_s$ and $g_j = h \circ f_t$ if and only if $i < j$. It follows that for every two homomorphisms $\hat{g}, \hat{g}' : \vec{G}_0 \to \vec{G}_n$ there exists an homomorphism $\hat{h} : \vec{G}_1 \to \vec{G}_n$ such that $\hat{g} = \hat{h} \circ \phi_s$ and $\hat{g} = \hat{h} \circ \phi_t$ if and only if the (uniquely determined) integers i, j such that $\hat{g} = \Phi(g_i)$ and $\hat{g}' = \Phi(g_j)$ are such that $i < j$.

Define the $n + 1$ tuples $\overline{x}^i = (x_0^i, \ldots, x_a^i)$ by $x_i^j = \Phi(g_j)(i)$. In other words, let $i \mapsto x_i^j$ be the homomorphism $\Phi(g_j)$. Then the above properties rewrites as

$$\vec{G}_n \models \eta(\overline{x}^i, \overline{x}^j) \quad \Longleftrightarrow \quad i < j.$$

It follows that $\mathcal{C}$ has the order property hence, by Theorem 2, is somewhere dense. $\qquad\square$

Remark 5. From a model theoretic point of view, the previous lemma is not surprising: As pointed out by a referee, since order indiscernibles in a stable class are always set indiscernibles (see e.g. [7]), the existence of a full embedding of the simplicial category Δ makes a class non-stable. The simplicial category Δ has several equivalent descriptions; the choice to describe Δ as the category of non-empty finite ordinals as objects (thought of as finite linear orders), and order preserving functions as morphisms, which displays the connection to order indiscernibles (see [19]), allowed us to include an easy elementary proof.

4. COMMENTS

1. Let us add few remarks putting the results of this work in a broader context. Representation of categories were first investigated in the special cases of groups, monoids, and small categories. This line of research directly relates to our main result.

For groups, the representřil can be done by graphs [11], and even by 3-regular graphs [27]. However this cannot be done by geometrically restricted graphs, like planar graphs or, more generally, by any proper minor closed class of graphs [4].

For monoids, the representation can be done by graphs with arbitrary girth (this is also possible by the above construction) but not by 3-regular or even k-regular graphs (for any fixed k). In fact Babai and Pultr [6] showed that any class of graphs which represents all finite monoids has to contain a subdivision of any complete graph. However, using large girth representations and using characterization of classes with bounded expansion [20, 24], one can easily see that finite monoids can be represented by graphs in a bounded expansion class $\mathcal{C}_0$. In particular, one can put $\mathcal{C}_0$ to be the class of all graphs of the form $G * (\vec{C}_{2n}, a, b)$, where $\vec{C}_{2n}$ is a circuit of length $2n$, where n is the order of G.

Consider small categories (in the theory of finite sets, that is finite categories). Let us enumerate all non-isomorphic small categories of graphs as $\mathcal{K}^1, \mathcal{K}^2, \ldots, \mathcal{K}^n, \ldots$.

Let $\mathcal{K}^i$ have objects $G_1^i, \ldots, G_{t(i)}^i$. The category $\mathcal{K}^i$ will be represented by oriented graphs of the form $G_j^i * (\vec{C}_{2N_i}, a, b)$, where $N_i \geq \sum_{j=1}^{t(i)} |V(G_j^i)|$ and $N_i < N_j$ whenever $i < j$. On sees easily that the class $\mathcal{C}_1$ of all such graphs $G_j^i *$

$(\vec{C}_{2N_i}, a, b)$ has bounded expansion: for any fixed integer d and any graph H, if the d-th subdivision of H is a subgraph of a graph in $\mathcal{C}_1$ then H is 2-degenerate with possibly finitely many exceptions (derived from graphs $G_j^i * (\vec{C}_{2N_i}, a, b)$ for small i). The class $\mathcal{C}_1$ represents all the small categories $\mathcal{K}^1, \mathcal{K}^2, \ldots, \mathcal{K}^n, \ldots$ by an application of Cayley-MacLane representation.

However to represent arbitrary categories (in the theory of finite sets) we have to jump over nowhere dense classes, right to somewhere dense classes. This descriptive complexity gap is surprising.

It would be interesting to find a more direct combinatorial proof of the fact that representing special categories leads to bounded subdivisions of arbitrarily large complete graphs. Such examples of groups and monoids were found in [4, 6].

2. In this context one should note that the representation of posets and thin categories can be achieved by oriented paths, trees, or outerplanar graphs [15, 16].

Let us summarize these facts in a schematic table.

	+	−
Posets	oriented trees, cycles, or paths	undirected bipartite
Groups	bounded degree	proper minor closed
Monoids	bounded expansion	proper topological minor closed
Small categories	bounded expansion	proper topological minor closed
Concrete categories	somewhere dense	nowhere dense

3. We restricted ourselves to the theory of finite sets (i.e. to finite graphs). The situation for infinite graphs and categories is less clear. On the other hand most examples of special algebraic universal categories are obtained from some basic examples (like the category of graphs) by first-order interpretation (like replacement operation in the above proof of Lemma 4). As the basic examples contain complete graphs of any size this leads then to p-subdivisions of large complete graphs. The main result of this paper shows that this is necessarily so.

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