CONANT'S GENERALISED METRIC SPACES ARE RAMSEY

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Abstract. We give Ramsey expansions of classes of generalised metric spaces where distances come from a linearly ordered commutative monoid. This complements results of Conant about the extension property for partial automorphisms and extends an earlier result of the first and the last author giving the Ramsey property of convexly ordered S-metric spaces. Unlike Conant's approach, our analysis does not require the monoid to be semi-archimedean.

Dedicated to old friend Norbert Sauer.

1. Introduction

Given $S \subseteq \mathbb{R}_{>0}$ (a subset of positive reals), an *S-metric space* is a metric space with all distances contained in $S \cup \{0\}$. The following 4-values condition characterises when the class of all finite S-metric spaces is closed for amalgamation (see Section 2 for definition of amalgamation):

Definition 1.1 (Delhommé, Laflamme, Pouzet, Sauer [7]). *A subset $S \subseteq \mathbb{R}_{>0}$ satisfies the* 4-values condition*, if for every $a, b, c, d \in S$, if there is some $x \in S$ such that the triangles with distances a–b–x and c–d–x satisfy the triangle inequality, then there is also $y \in S$ such that the triangles with distances a–c–y and b–d–y satisfy the triangle inequality.*

The 4-value condition means that the amalgamation of triangles exists (see Figure 1). It can be equivalently described by means of the following operation:

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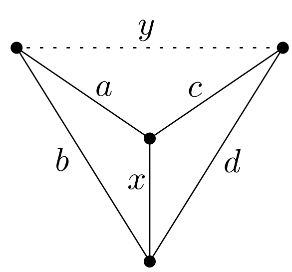

Figure 1. The 4-values condition.

Definition 1.2. *Given a subset S of positive reals and $a, b \in S$, denote by* $a \oplus_S b = \sup\{x \in S; x \leq a + b\}$.

Theorem 1.3 (Sauer [18])**.** *A subset S of the positive reals satisfies the 4-values condition if and only if the operation $\oplus_S$ is associative.*

Sauer [17] used the equivalence above to determine those S such that there exists an (ultra)homogeneous S-metric space (i.e. the S-Urysohn metric space). In [13] the first and third author proved:

Theorem 1.4 (Hubička, Nešetřil [13])**.** *Given set S of positive reals the following four statements are equivalent:*

(1) The class of all finite S-metric spaces is an amalgamation class.
(2) S satisfies 4-values condition.
(3) $\oplus_S$ is associative.
(4) The class of all finite S-metric spaces has a precompact Ramsey expansion.

This generalises the earlier work on Ramsey property for metric spaces [14, 8, 16]. In this paper we further develop this line of research and show similar results in the context of generalised metric spaces where distance sets form a monoid as defined by Conant [6] and show that the class of all such generalised metric spaces is Ramsey when enriched by a convex linear ordering (see e.g. [16]).

Let us recall the key notion of a Ramsey class first. For structures $\mathbf{A}, \mathbf{B}$ denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all substructures of $\mathbf{B}$, which are isomorphic to $\mathbf{A}$ (see Section 2 for definition of structure and substructure). Using this notation the definition of a Ramsey class gets the following form:

Definition 1.5. *A class $\mathcal{C}$ is a* Ramsey class *if for every two objects $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$ and for every positive integer k there exists a structure $\mathbf{C}$ in $\mathcal{C}$ such that the following holds: For every partition $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there exists an $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition.*

It is usual to shorten the last part of the definition to $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_k$.

The following is the main result of this paper which complements results of Conant [6] about the extension property for partial automorphisms (EPPA). Conant generalized results of Solecki [19] and Vershik [21] about metric spaces. Our work extends earlier Theorem 1.4 in a similar manner.

Theorem 1.6. *For every distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ the class $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ of all convexly ordered finite $\mathfrak{M}$-metric spaces is Ramsey.*

To define a distance monoid we first recall a standard definition. The definition of convex order will be given in Definition 6.10.

A *commutative monoid* is a triple $\mathfrak{M} = (M, \oplus, \preceq, 0)$ where M is a set containing 0, $\oplus$ is an associative and commutative binary operation on M and 0 is the identity element of $\oplus$.

The following definition of a distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ was given by Conant [6]:

Definition 1.7. *A structure* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *is a* distance monoid *if*

(1) $(M, \oplus, 0)$ *is a commutative monoid with identity* 0*;*
(2) $(M, \preceq, 0)$ *is a linear order with least element* 0*;*
(3) For all $a, b, c, d \in M$ *it holds that if* $a \preceq c$ *and* $b \preceq d$*, then* $a \oplus b \preceq c \oplus d$ *(*$\oplus$ *is monotonous in* $\preceq$*).*

In other words, distance monoid is a (linearly) positively ordered monoid, see [22]. For every distance monoid $\mathfrak{M}$, one can define a $\mathfrak{M}$-metric space.

Definition 1.8. *Suppose* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *is a distance monoid. Given a set* A *and a function* $d : A \times A \to M$*, we call* (A, d) *an* $\mathfrak{M}$-metric space *if*

(1) for all $a, b \in A$*,* $d(a, b) = 0$ *if and only if* $a = b$*;*
(2) for all $a, b \in A$*,* $d(a, b) = d(b, a)$*;*
(3) for all $a, b, c \in A$*,* $d(a, c) \preceq d(a, b) \oplus d(b, c)$*.*

Given a distance monoid $\mathfrak{M}$*, we let* $\mathcal{M}_{\mathfrak{M}}$ *denote the class of finite* $\mathfrak{M}$*-metric spaces.*

Example 1.9. *The following are distance monoids:*

(1) Given set S *of non-negative reals containing 0 the structure* $\mathfrak{M}_S = (S, \oplus_S, \leq, 0)$ *(recall Definition 1.2), where the order* $\leq$ *is the linear order of reals, forms a distance monoid if and only if operation* $\oplus_S$ *is associative and thus* S *satisfies the 4-values condition.*
(2) Consider the set of nonnegative real numbers extended by infinitesimal elements, i.e. $R^* = \left\{a + b \cdot \mathrm{dx} \mid a, b \in \mathbb{R}_0^+\right\}$ *with piece-wise addition* $+$ *and order* $\preceq$ *given by the standard order of reals and* $\mathrm{dx} \prec a$ *for every positive real number* a*. Then* $(R^*, +, \preceq, 0)$ *is also a distance monoid.*
(3) The ultrametric $([n], \max, \leq, 0)$*, where* $[n] = \{0, 1, \ldots, n-1\}$ *and* $\leq$ *is the linear order of integers is a distance monoid.*

Given $n \geq 0$ and $r \in M$ we denote by $n \times r$ a sequence $r \oplus r \oplus \cdots \oplus r$ of length n.

Definition 1.10. *A distance monoid* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *is* archimedean *if, for all* $r, s \in M$*,* $r, s \neq 0$*, there exists some integer* $n > 0$ *such that* $s \preceq n \times r$*.*

Example 1.11. *Consider the reals extended by infinitesimals as in Example 1.9 (2). This monoid is not archimedean, because* $n \times \mathrm{dx} \prec b$ *for every positive real* b*, every integer* n *and infinitesimal* dx*.*

In Section 2 we briefly introduce necessary model-theoretic background. In Section 3 we review Ramsey classes defined by means of forbidden subconfigurations in the setting of [13]. Although we deal with metric spaces and thus binary systems, it is useful to formulate it in the context of structures involving both relations and functions which will be used in the proof of our main result. In Section 4 we discuss simple algorithm completing graphs to metric spaces which is essential for our approach. In Section 5 we show that the class of finite ordered $\mathfrak{M}$-metric spaces is Ramsey for every archimedean monoid $\mathfrak{M}$. Finally, in Section 6 we prove the main result and in Secion 7 we discuss future directions of research.

2. Preliminaries

We now review some standard model-theoretic notions of structures with relations and functions (see e.g. [11]) with a small variation that our functions will be partial and symmetric. We follow [15].

Let $L = L_{\mathcal{R}} \cup L_{\mathcal{F}}$ be a language involving relational symbols $R \in L_{\mathcal{R}}$ and function symbols $F \in L_{\mathcal{F}}$ each having associated arities denoted by $a(R) > 0$ for relations and $d(F) > 0, r(F) > 0$ for functions. An *L-structure* is a structure $\mathbf{A}$ with *vertex set* A, functions $F_{\mathbf{A}} : \mathrm{Dom}(F_{\mathbf{A}}) \to \binom{A}{r(F)}$, $\mathrm{Dom}(F_{\mathbf{A}}) \subseteq A^{d(F)}$ for $F \in L_F$ and relations $R_{\mathbf{A}} \subseteq A^{a(R)}$ for $R \in L_R$. (Note that by $\binom{A}{r(F)}$ we denote, as it is usual in this context, the set of all $r(F)$-element subsets of A.) Set $\mathrm{Dom}(F_{\mathbf{A}})$ is called the *domain* of function F in $\mathbf{A}$.

Note also that we have chosen to have the range of the function symbols to be the set of subsets (not tuples). This is motivated by [9] where we deal with (Hrushovski) extension properties and we need a “symmetric” range. However from the point of view of Ramsey theory this is not an important issue.

The language is usually fixed and understood from the context (and it is in most cases denoted by L). If set A is finite we call $\mathbf{A}$ a *L-structure.* We consider only structures with countably many vertices. If language L contains no function symbols, we call L a *relational language* and an L-structure is also called a *relational L-structure.* Every function symbol F such that $d(F) = 1$ is a *unary function.* Unary relation is of course just defining a subset of elements of A. All functions used in this paper are unary.

A *homomorphism* $f : \mathbf{A} \to \mathbf{B}$ is a mapping $f : A \to B$ satisfying for every $R \in L_{\mathcal{R}}$ and for every $F \in L_{\mathcal{F}}$ the following two statements:

(a) $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$, and,

(b) $f(\mathrm{Dom}(F_{\mathbf{A}})) \subseteq \mathrm{Dom}(F_{\mathbf{B}})$ and $f(F_{\mathbf{A}}(x_1, x_2, \ldots, x_{d(F)})) = F_{\mathbf{B}}(f(x_1), f(x_2), \ldots, f(x_{d(F)}))$ for every $(x_1, x_2, \ldots, x_{d(F)}) \in \mathrm{Dom}(F_{\mathbf{A}})$.

For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x); x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure.

If f is injective, then f is called a *monomorphism.* A monomorphism is called an *embedding* if for every $R \in L_{\mathcal{R}}$ and $F \in L_{\mathcal{F}}$ the following holds:

(a) $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$, and,

(b) $(x_1, x_2, \dots, x_{d(F)}) \in \mathrm{Dom}(F_{\mathbf{A}}) \iff (f(x_1), f(x_2), \dots, f(x_{d(F)})) \in \mathrm{Dom}(F_{\mathbf{B}})$.

If f is an embedding which is an inclusion then $\mathbf{A}$ is a *substructure* (or *subobject*) of $\mathbf{B}$. Observe that for structures with functions it does not hold that every choice of $A \subseteq B$ induces a substructure of $\mathbf{B}$.

For an embedding $f : \mathbf{A} \to \mathbf{B}$ we say that $\mathbf{A}$ is *isomorphic* to $f(\mathbf{A})$ and $f(\mathbf{A})$ is also called a *copy* of $\mathbf{A}$ in $\mathbf{B}$. Thus $\binom{\mathbf{B}}{\mathbf{A}}$ is defined as the set of all copies of $\mathbf{A}$ in $\mathbf{B}$. Finally, $\mathrm{Str}(L)$ denotes the class of all finite L-structures and all their embeddings.

Let $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ be structures with α_1 an embedding of $\mathbf{A}$ into $\mathbf{B}_1$ and α_2 an embedding of $\mathbf{A}$ into $\mathbf{B}_2$, then every structure $\mathbf{C}$ together with embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \to \mathbf{C}$ satisfying $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *over* $\mathbf{A}$ *with respect to* α_1 *and* α_2. We will call $\mathbf{C}$ simply an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (as in the most cases α_1, α_2 and β_1, β_2 can be chosen to be inclusion embeddings).

Amalgamation is *strong* if $C = \beta_1(B_1) \cup \beta_2(B_2)$ and moreover $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$. Strong amalgamation is *free* there are no tuples in any relations of $\mathbf{C}$ and $\mathrm{Dom}(F_{\mathbf{C}})$, $F \in L_{\mathcal{F}}$, using both vertices of $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$. An *amalgamation class* is a class $\mathcal{K}$ of finite structures satisfying the following three conditions:

(1) *Hereditary property:* For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;
(2) *Joint embedding property:* For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C}$ contains both $\mathbf{A}$ and $\mathbf{B}$ as substructures;
(3) *Amalgamation property:* For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

If the $\mathbf{C}$ in the amalgamation property can always be chosen as the free amalgamation, then $\mathcal{K}$ is *free amalgamation class.*

3. Previous work — multiamalgamation

We now refine amalgamation classes. Our aim is to describe strong sufficient criteria for Ramsey classes. In this paper we follow [13].

For L containing a binary relation $R^{\leq}$ we denote by $\overrightarrow{\mathrm{Str}}(L)$ the class of all finite L-structures $\mathbf{A} \in \mathrm{Str}(L)$ where the set A is linearly ordered by the relation $R^{\leq}$. $\overrightarrow{\mathrm{Str}}(L)$ is of course considered with all monotone (i.e. order preserving) embeddings.

In this setting, we develop a generalised notion of amalgamation which will serve as a useful tool for the construction of Ramsey objects. As schematically depicted in Figure 2, Ramsey objects are a result of amalgamation of multiple copies of a given structure which are all performed at once. In a non-trivial class this leads to many problems. We split the amalgamation

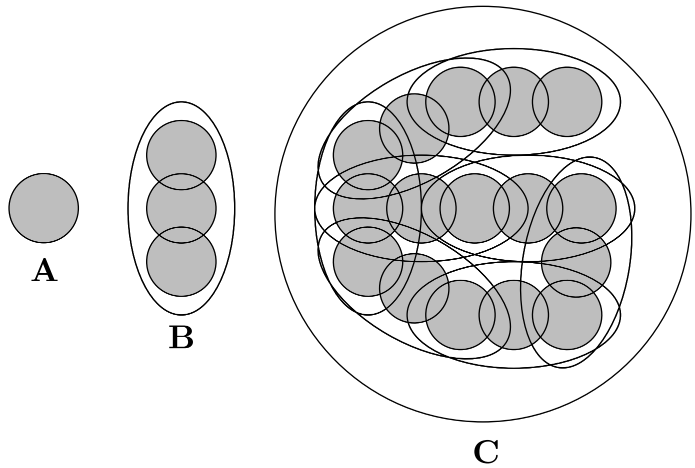

Figure 2. Construction of a Ramsey object by multiamalgamation.

into two steps—the construction of (up to isomorphism unique) free amalgamation (which yields an incomplete or "partial" structure) followed then by a completion. Formally this is done as follows:

Definition 3.1. *An L-structure* $\mathbf{A}$ *is* irreducible *if* $\mathbf{A}$ *is not a free amalgamation of two proper substructures of* $\mathbf{A}$*.*

Thus the irreducibility is meant with respect to the free amalgamation. The irreducible structures are our building blocks. Moreover in structural Ramsey theory we are fortunate that most structures are (or may be interpreted as) irreducible. And in the most interesting case, the structures may be completed to irreducible structures. This will be introduced now by means of the following variant of the homomorphism notion.

Definition 3.2. *A homomorphism* $f : \mathbf{A} \to \mathbf{B}$ *is a* homomorphism-embedding *if* f *restricted to any irreducible substructure of* $\mathbf{A}$ *is an embedding to* $\mathbf{B}$*.*

While for (undirected) graphs the homomorphism and homomorphism-embedding coincide, for structures they differ. For example any homomorphism-embedding of the Fano plane into a hypergraph is actually an embedding.

Definition 3.3. *Let* $\mathbf{C}$ *be a structure. An irreducible structure* $\mathbf{C}'$ *is a* completion *of* $\mathbf{C}$ *if there exists a homomorphism-embedding* $\mathbf{C} \to \mathbf{C}'$*.*

Let $\mathbf{B}$ *be an irreducible substructure of* $\mathbf{C}$*. We say that irreducible structure* $\mathbf{C}'$ *is a* completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$ *if there exists a function* $f : C \to C'$ *such that for every* $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ *the function* f *restricted to* $\widetilde{B}$ *is an embedding of* $\widetilde{\mathbf{B}}$ *to* $\mathbf{C}'$*.*

We now state all necessary conditions for main result of [13] which will be used subsequentely.

Definition 3.4. *Let L be a language, $\mathcal{R}$ be a Ramsey class of finite irreducible L-structures. We say that a subclass $\mathcal{K}$ of $\mathcal{R}$ is an* $\mathcal{R}$-multiamalgamation class *if the following conditions are satisfied:*

(1) Hereditary property: *For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$.*

(2) Strong amalgamation property: *For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which contains a strong amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2 as a substructure.*

(3) Locally finite completion property: *Let $\mathbf{B} \in \mathcal{K}$ and $\mathbf{C}_0 \in \mathcal{R}$. Then there exists $n = n(\mathbf{B}, \mathbf{C}_0)$ such that if a closed L-structure $\mathbf{C}$ satisfies the following:*

   (a) there is a homomorphism-embedding from $\mathbf{C}$ to $\mathbf{C}_0$, (in other words, $\mathbf{C}_0$ is a completion of $\mathbf{C}$), and,

   (b) every substructure of $\mathbf{C}$ with at most n vertices has a completion to some structure in $\mathcal{K}$.

   Then there exists $\mathbf{C}' \in \mathcal{K}$ that is a completion of $\mathbf{C}$ with respect to copies of $\mathbf{B}$.

We can now state the main result of [13] as:

Theorem 3.5 (Hubička, Nešetřil [13])**.** *Every $\mathcal{R}$-multiamalgamation class $\mathcal{K}$ is Ramsey.*

The proof of this result is not easy and involves interplay of several key constructions of structural Ramsey theory, particularly Partite Lemma and Partite Construction (see [13] for details). Theorem 3.6. We will also make use of the following recent strengthening of Nešetřil-Rödl Theorem [15]:

Theorem 3.6 (Evans, Hubička, Nešetřil [9])**.** *Let L be a language (involving relational symbols and partial functions) and let $\mathcal{K}$ be a free amalgamation class of L-structures. Then $\overrightarrow{\mathcal{K}}$ is a Ramsey class.*

4. Shortest path completion

We first show some basic facts about completion to $\mathfrak{M}$-metric spaces. This is similar to [13] and also analysis given in [6] proceeds similarly.

Given a distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ we interpret an $\mathfrak{M}$-metric space as a relational structure $\mathbf{A}$ in the language $L_{\mathfrak{M}}$ with (possibly infinitely many) binary relations R^s, $s \in M^{\succ 0}$, where we put, for every $u \neq v \in A$, $(u, v) \in R^\ell_{\mathbf{A}}$ if and only if $d(u, v) = \ell$. We do not explicitly represent that $d(u, u) = 0$ (i.e. no loops are added).

Definition 4.1. *Every $L_{\mathfrak{M}}$-structure where all relations are symmetric and irreflexive and every pair of vertices is in at most one relation is an* $\mathfrak{M}$-graph *(a graph with edges labelled by $M^{\succ 0}$).*

Every non-induced substructure of an $\mathfrak{M}$-metric space (interpreted as $L_{\mathfrak{M}}$-structure) such that all relations are symmetric is an $\mathfrak{M}$-metric graph.

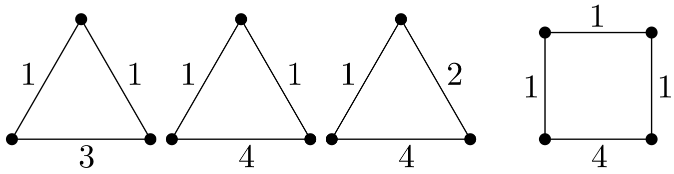

FIGURE 3. All non-metric cycles with distances $1, 2, 3$ and 4.

Every $\mathfrak{M}$*-graph that is not* $\mathfrak{M}$*-metric is a* non-$\mathfrak{M}$-metric graph.

Observe that $\mathfrak{M}$-metric graphs are precisely those structures which have a strong completion to an $\mathfrak{M}$-metric space in the sense of Definition 3.3.

For $\mathfrak{M}$-graph $\mathbf{A}$ we will write $d(u, v) = \ell$ if $(a, b) \in R^\ell_\mathbf{A}$. Value of $d(u, v)$ is undefined otherwise.

In the language of $\mathfrak{M}$-graphs we will use the following variants of standard graph-theoretic notions. Given $\mathfrak{M}$-graph $\mathbf{A}$ the *walk* from u to v is any sequence of vertices $u = v_1, v_2, \dots, v_n = v \in \mathbf{A}$ such that $d(v_i, v_{i+1})$ is defined for every $1 \leq i < n$. If the sequence contains no repeated vertices, it is a *path*. The $\mathfrak{M}$-length of walk (or path) is $d(v_1, v_2) \oplus d(v_2, v_3) \oplus \cdots \oplus d(v_{n-1}, v_n)$. Given vertices u and v the *shortest path from* u *to* v is any path from u to v such that there is no other path from u to v of strictly smaller $\mathfrak{M}$-length (in the order $\preceq$). We say that $\mathbf{A}$ is *connected* if there exists a path from u to v for every choice of $u \neq v \in A$.

Definition 4.2 (Shortest path completion). *Let* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *be a distance monoid and* $\mathbf{G} = (G, d)$ *be a (finite) connected* $\mathfrak{M}$*-metric graph. For every* $u, v \in A$ *define* $d'(u, v)$ *to be the minimum of the* $\mathfrak{M}$*-lengths of all walks from* u *to* v *in* $\mathbf{G}$*. Then we call the complete* $\mathfrak{M}$*-metric graph* $\mathbf{A} = (G, d')$ *the* shortest path completion *of* $\mathbf{G}$.

Given an $\mathfrak{M}$*-metric graph we also denote by* $\mathcal{W}(u, v)$ *the shortest path connecting* u *to* v *such that its* $\mathfrak{M}$*-length is* $d'(u, v)$.

The following is the main result of this section.

Proposition 4.3. *Let* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *be a distance monoid and* $\mathbf{G}$ *be a finite* $\mathfrak{M}$*-graph.*

1. *If* $\mathbf{G}$ *is connected and* $\mathfrak{M}$*-metric, then its shortest path completion* $\mathbf{A}$ *is an* $\mathfrak{M}$*-metric space.*
2. $\mathbf{G}$ *is* $\mathfrak{M}$*-metric if and only if it contains no homomorphic image of a non-*$\mathfrak{M}$*-metric cycle* $\mathbf{C}$ *(a cycle* $v_1, v_2, \dots, v_n$ *such that* $d(v_1, v_n) \succ d(v_1, v_2) \oplus \cdots \oplus d(v_{n-1}, v_n)$*).*
3. $\mathbf{G}$ *contains a homomorphic image of a non-*$\mathfrak{M}$*-metric cycle if and only if it contains a non-*$\mathfrak{M}$*-metric cycle as an induced substructure (i.e. a monomorphic image).*
4. $\mathcal{M}_\mathfrak{M}$ *is an amalgamation class closed for strong amalgamation.*

Example 4.4. *Consider monoid* $\mathfrak{M}_S$ *as in Example 1.9 (1) for* $S = \{1, 2, 3, 4\}$*. Figure 3 depicts all non-metric cycles which prevent completion of* $\mathfrak{M}$*-graph to* $\mathfrak{M}$*-metric space given by Proposition 4.3 (3).*

Proof. All three statements are consequences of associativity of $\oplus$.

(1). Assume that $\mathbf{G}$ is $\mathfrak{M}$-metric. First we show that the completion described will give an $\mathfrak{M}$-metric space by verifying that d' satisfies the triangle inequality. Assume, to the contrary, the existence of vertices u, v, w such that $d'(u,v) \succ d'(u,w) \oplus d'(w,v)$. Combine the walks $\mathcal{W}(u,w)$ and $\mathcal{W}(w,v)$ to get a walk from u to v in $\mathbf{G}$ whose length is $d'(u,w) \oplus d'(w,v)$. It follows that $d'(u,v) \preceq d'(u,w) \oplus d'(w,v)$ which is a contradiction.

We have shown that d' forms an $\mathfrak{M}$-metric space on vertices of $\mathbf{G}$ but it still needs to be checked that $d_{\mathbf{G}}(u,v) = d'(u,v)$ whenever $d_{\mathbf{G}}(u,v)$ is defined. We show a stronger claim: if $\mathbf{B}$ is a completion of $\mathbf{G}$ to an $\mathfrak{M}$-metric space then $d_{\mathbf{B}}(u,v) \preceq d'(u,v)$ for every $u \neq v \in \mathbf{G}$.

Suppose, for a contradiction, that there are vertices $u \neq v \in \mathbf{G}$ such that $d_{\mathbf{B}}(u,v) \succ d'(u,v)$. By definition of d', there is a path $\mathcal{W}(u,v)$ in $\mathbf{G}$ with $d'(u,v)$ being its length and then $d_{\mathbf{B}}(u,v) \succ d'(u,v)$ contradicts $\mathbf{B}$ being a $\mathfrak{M}$-metric space.

(2). We first show (2) for connected $\mathfrak{M}$-graphs only. Assume that $\mathbf{G} = (G,d)$ is non-$\mathfrak{M}$-metric and let $\mathbf{A} = (G,d')$ be its shortest path completion. $\mathbf{G}$ being non-metric means that there are vertices $u \neq v \in G$ with $d'(u,v) \prec d(u,v)$. But that means that the $\mathfrak{M}$-length of $W(u,v)$ is strictly less than $d(u,v)$, hence this path together with the edge u,v forms a non-$\mathfrak{M}$-metric cycle.

(3). Again first consider connected $\mathfrak{M}$-graphs only. One implication is trivial. The other follows from $\oplus$ being monotonous with respect to $\preceq$ – it is enough to take the minimal subcycle of the homomorphic image of the non-$\mathfrak{M}$-metric cycle containing the edge v_1v_n (see (2)).

Now it is easy to see that both (2) and (3) also holds for $\mathfrak{M}$-graphs that are not connected, because every such $\mathbf{G}$ can be turned into connected one by adding new edges connecting individual components without introducing new cycles.

(4). Given $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{M}_{\mathfrak{M}}$ it is easy to see that the free amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ contains no embedding of any non-metric cycle as described in the previous paragraph. □

5. Archimedean monoids

In this section we use the machinery introduced in Section 3 to show that for an archimedean monoid $\mathfrak{M}$ the class $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ is Ramsey. This is supposed to serve as a warm-up for Section 6, where we deal with general distance monoids and the means are considerably more difficult.

Lemma 5.1. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be an archimedean distance monoid. Then for every $a, b \in M$, $b \succ 0$ either $a \oplus b \succ a$ or a is the maximal element of $\mathfrak{M}$.*

Proof. Assume the contrary and consider a, b such that $a \oplus b = a$ and a is not the maximal element of $\mathfrak{M}$. In this case also $a \oplus (n \times b) = a$ for every n. Because a is not maximal element there exists $c \in \mathfrak{M}$ such that $a \prec c$. Because $\mathfrak{M}$ is archimedean we however know that there is n such that $n \times b \succeq c \succ a$. A contradiction with monotonicity of $\oplus$. □

The following lemma is the basic tool used to show local finiteness condition needed by Theorem 3.5.

Lemma 5.2. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be an archimedean distance monoid and let $S \subseteq M$ be finite. Then there exists $n = n(S)$ such that for every non-$\mathfrak{M}$-metric cycle $\mathbf{C}$ such that all distances in $\mathbf{C}$ are from S it holds that $\mathbf{C}$ has at most n vertices.*

Proof. Because $\mathfrak{M}$ is archimedean, for every $a, b \in \mathfrak{M}$ there exists a smallest $m = m(a, b)$ such that $a \preceq m \times b$. Let $n = \max\{m(a, b) \mid a, b \in S\}$. Then by Lemma 5.1 it follows for every $a_1, a_2, \ldots, a_n \in S$ that $a_1 \oplus a_2 \oplus \cdots a_n$ is at least the largest element in S and hence if $\mathbf{C}$ has at least $n+1$ vertices, it cannot be non-$\mathfrak{M}$-metric. □

Now we are ready to show that the class of all finite $\mathfrak{M}$-metric spaces has Ramsey expansion.

Theorem 5.3. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be an archimedean distance monoid. Then the class of all finite $\mathfrak{M}$-metric spaces with free, i.e. arbitrary, ordering of vertices, $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$, is a Ramsey class.*

Proof. Let $\mathbf{C}_0$ be an arbitrary finite linearly ordered $\mathfrak{M}$-graph. We will show that there exists a $n = n(\mathbf{C}_0)$ such that every $\mathfrak{M}$-graph $\mathbf{C}$ with additional binary relation $\leq_{\mathbf{C}}$ and a homomorphism-embedding to $\mathbf{C}_0$ is $\mathfrak{M}$-metric, given that every substructure of $\mathbf{C}$ on at most n vertices is $\mathfrak{M}$-metric, thereby checking the conditions of Theorem 3.5 (the strong amalgamation property is given by Proposition 4.3 (4) and remaining assumptions are trivial).

Let S be the set of distances which appear in $\mathbf{C}_0$. As $\mathbf{C}_0$ is finite, S is clearly finite, too. Take $n = n(S)$ from Lemma 5.2. Then every non-metric cycle has at most n vertices. And as every non-$\mathfrak{M}$-metric graph contains a non-$\mathfrak{M}$-metric cycle by Proposition 4.3 (3), the statement follows. □

6. General distance monoids

In this section we generalise the construction to distance monoids in full generality. In particular, unlike [6] we do not need the notion of semi-archimedean monoids. (A distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ is *semi-archimedean* if, for all $r, s \in M$, $r, s \succ 0$ if $n \times r \prec s$ for all $n > 0$ then $r \oplus s = s$.)

The main difficulties in generalizing Theorem 5.3 come from the fact that there is no direct equivalent to Lemma 5.2. Consider, for example, spaces with distances 1 and 3. Every such metric space consists of a disjoint union of balls of diameter 1 separated by distance 3. Every cycle having one distance 3 and rest of distances 1 is forbidden regardless of number of its vertices.

To overcome this problem we need to precisely characterise definable equivalences in $\mathfrak{M}$-metric spaces and represent them by means of artificial vertices and functions. In the language of model theory, we are going to eliminate imaginaries, see [13] for details.

6.1. **Blocks and block equivalences.** As we will show, the definable equivalences are related to archimedean submonoids of $\mathfrak{M}$-metric spaces. The following is a generalization of definition by Sauer [17].

Definition 6.1. *Given a distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$, a* block $\mathcal{B}$ *of $\mathfrak{M}$ is every subset of M such that either*

(1) $\mathcal{B} = \{0\}$, or
(2) $0 \notin \mathcal{B}$ and $\{0\} \cup \mathcal{B}$ induces a maximal archimedean submonoid of $\mathfrak{M}$.

Given a block $\mathcal{B}$ we will denote by $\mathfrak{M}_{\mathcal{B}}$ the archimedean submonoid induced by it.

The basic properties of blocks can be summarized as follows.

Lemma 6.2. *Given a distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ it holds that:*

(1) For every $a \in M$ there exists a unique block $\mathcal{B}_a$ containing a.
(2) Let $a, b \in \mathfrak{M}$. If there exist m, n such that $m \times a \succeq b$ and $n \times b \succeq a$, then a, b are in the same block.

Proof. (1) Let

$$\mathcal{B}_a = \{b \in \mathfrak{M} \mid (\exists n)(n \times a \succeq b) \wedge (\exists n)(n \times b \succeq a)\}.$$

It is easy to check that $\mathfrak{M}_{\mathcal{B}_a} = (\mathcal{B}_a \cup \{0\}, \oplus, 0, \preceq)$ is an archimedean submonoid of $\mathfrak{M}$. The maximality and uniqueness follows from the fact that no $b \in M \setminus (\mathcal{B}_a \cup \{0\})$ can be in the same archimedean submonoid as a.

(2) follows from the proof of (1). □

Given a distance monoid $\mathfrak{M} = (M, \oplus, \preceq, 0)$ and $a \in M$, we will always denote by $\mathcal{B}_a$ the unique block of $\mathfrak{M}$ containing a given by Lemma 6.2 (1).

Lemma 6.3. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be a distance monoid. Then there are no two blocks $\mathcal{B}, \mathcal{B}'$ and elements $a, c \in \mathcal{B}, b \in \mathcal{B}'$ with $a \prec b \prec c$.*

Proof. Suppose for a contradiction that the statement is not true. As $a, c \in \mathcal{B}$, there is n such that $n \times a \succeq c$. But then also $n \times a \succeq b$ and hence by Lemma 6.2 (2) a and b are in the same block, which is a contradiction. □

This means that in the order $\preceq$, blocks form intervals. And this motivates the following definition:

Definition 6.4. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be a distance monoid. By same symbol $\preceq$ we denote the on blocks of $\mathfrak{M}$ putting $\mathcal{B} \preceq \mathcal{B}'$ if for every $a \in \mathcal{B}, b \in \mathcal{B}'$ it holds that $a \preceq b$.*

By Lemma 6.3 $\preceq$ is a linear order of blocks of $\mathfrak{M}$.

Definition 6.5. *Let* $\mathbf{A}$ *be an* $\mathfrak{M}$*-metric space and* $\mathcal{B}$ *block of* $\mathfrak{M}$*.*

(1) A block equivalence $\sim_{\mathcal{B}}$ *on vertices of* $\mathbf{A}$ *is given by* $u \sim_{\mathcal{B}} v$ *whenever there exists* $a \in \mathcal{B}$ *such that* $d(u, v) \preceq a$*.*

(2) A ball of diameter $\mathcal{B}$ *in* $\mathbf{A}$ *is any equivalence class of* $\sim_{\mathcal{B}}$ *in* $\mathbf{A}$*.*

Note that a block $\mathcal{B}$ does not need to contain maximal element.

To verify that for every block $\mathcal{B}$ the relation $\sim_{\mathcal{B}}$ is indeed an equivalence relation it suffices to check transitivity. Given a triangle with distances a, b, c, if there exists $a' \in \mathcal{B}$ such that $a \preceq a'$ and $b' \in \mathcal{B}$ such that $b \preceq b'$ it also holds that $c \preceq a \oplus b \preceq a' \oplus b' \in \mathcal{B}$.

6.2. **Important and unimportant summands.** This rather technical part is the key to obtaining a locally finite description of $\mathcal{M}_{\mathfrak{M}}$ (needed for Theorem 3.5).

Given $\mathfrak{M} = (M, \oplus, \preceq, 0)$ and $S \subseteq M$ will denote by $S^{\oplus}$ the set of all values which can be obtained as nonempty sums of values in S. (Thus $S^{\oplus} \cup \{0\}$ forms the submonoid of $\mathfrak{M}$ generated by S.)

Blocks of a monoid may be infinite and may not contain a maximal element which would be useful in our arguments (those maximal elements are refered to as jump numbers in [13]). For a fixed finite $S \subseteq M$ we seek a sufficient approximation $\mathrm{mus}(\mathcal{B}, S)$ of $\max(\mathcal{B})$ given by the following lemma. The name mus means *maximum useful* distance (with respect to $\mathcal{B}$ and S).

Lemma 6.6. *Let* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *be a distance monoid with finitely many blocks and* $S \subseteq M$ *be a finite subset of* M*. Then for every nonzero block* $\mathcal{B}$ *of* $\mathfrak{M}$ *there is a distance* $\mathrm{mus}(\mathcal{B}, S) \in \mathcal{B}$ *such that for every* $\ell \in S$ *and* $e \in S^{\oplus}$ *one of the following holds:*

(1) $e \oplus \mathrm{mus}(\mathcal{B}, S) \succeq \ell$*, or*

(2) $e \oplus b \prec \ell$ *for every* $b \in \mathcal{B}$ *(and thus also for every* $b \in \mathcal{B}'$*, where* $\mathcal{B}' \preceq \mathcal{B}$*).*

Example 6.7. *It is always possible to put* $\mathrm{mus}(\mathcal{B}, S)$ *to be the maximal element of* $\mathcal{B}$ *if it exists. The choice may be more difficult for blocks with no maximal element. To clarify this consider the monoid with infinitesimals given in Example 1.9 (2). This monoid has three blocks.* $\mathcal{B}_0 = \{0\}$*,* $\mathcal{B}_1$ *consists of infinitesimals and* $\mathcal{B}_2$ *of all remaining values. For* $S = \{\mathrm{dx}, 1, 2 + 3\mathrm{dx}\}$ *we can put* $\mathrm{mus}(\mathcal{B}_1, S) = 3\mathrm{dx}$ *and* $\mathrm{mus}(\mathcal{B}_2, S) = 1 + 3\mathrm{dx}$*. Note that* $\mathrm{mus}(\mathcal{B}_1, S) = 3\mathrm{dx} \notin S$*.*

Proof of Lemma 6.6. Let S be a fixed finite subset of M. Enumerate blocks of $\mathfrak{M}$ as $\mathcal{B}_1 \succeq \mathcal{B}_2 \succeq \cdots \succeq \mathcal{B}_p$.

Given a block $\mathcal{B}$ and distances $\ell, e \in M$, define $f(\mathcal{B}, \ell, e)$ to be some (for example the smallest, if it exists) $a \in \mathcal{B}$ such that $\ell \preceq e \oplus a$ or zero if $\ell \succ e \oplus a$ for all $a \in \mathcal{B}$. Further, given a block $\mathcal{B}$ and a distance $e \in \mathcal{B}$, define

$$X(\mathcal{B}, e) = \{a \in \mathcal{B} \cup \{0\} \mid a \preceq e \text{ and } \exists b_1, \ldots, b_m \in \mathcal{B} \cap S : a = b_1 \oplus \cdots \oplus b_m\}.$$

Observe that $X(\mathcal{B}, e)$ is finite for any choice of $\mathcal{B}$ and e: As $\mathfrak{M}_\mathcal{B}$ is archimedean, we have for each $b \in \mathcal{B} \cap S$ some finite $n(b)$ such that $n(b) \times b \succeq e$. As $\mathcal{B} \cap S$ is also finite and $\preceq$ is a congruence for $\oplus$, $X(\mathcal{B}, e)$ is finite.

Now, for a given $\ell \in S$, we define by induction the sets $X_i(\ell)$ and distances $d_i(\ell)$:

$$d_i(\ell) = \begin{cases} \text{an arbitrary } a \in \mathcal{B}_i & \text{if } \mathcal{B}_i \succ \mathcal{B}_\ell, \\ \ell & \text{if } \mathcal{B}_i = \mathcal{B}_\ell, \\ \max_{e \in X_{i-1}} f(\mathcal{B}_i, \ell, e) & \text{if } \mathcal{B}_i \prec \mathcal{B}_\ell; \end{cases}$$

$$X_i(\ell) = \begin{cases} \emptyset & \text{if } \mathcal{B}_i \succ \mathcal{B}_\ell, \\ X_{i-1} \oplus X(\mathcal{B}_i, d_i) & \text{if } \mathcal{B}_i \preceq \mathcal{B}_\ell, \end{cases}$$

where $A \oplus B = \{a \oplus b \mid a \in A, b \in B\}$.

Claim 6.8. *The following two statements are true for every $1 \leq i \leq p$ and $\ell \in S$:*

(1) For every $e \in S^\oplus$ either $e \oplus d_i(\ell) \succeq \ell$, or $e \oplus b \prec \ell$ for every $b \in \mathcal{B}_i$.
(2) Let $e \in S^\oplus \cap \bigcup_{j \leq i} \mathcal{B}_j$ be a distance such that $e \prec \ell$ and there is $b \in \mathcal{B}_{i+1}$ with $e \oplus b \succeq \ell$. Then $e \in X_i(\ell)$.

By (1) Lemma 6.6 follows. Put

$$\mathrm{mus}(\mathcal{B}_i, S) = \max_{\ell \in S} d_i(\ell)$$

and as a special case, if for some i we would have $\mathrm{mus}(\mathcal{B}_i, S) = 0$ choose $\mathrm{mus}(\mathcal{B}_i, S) \in \mathcal{B}_i$ arbitrarily.

Thus it remains to prove Claim 6.8. We will proceed by induction. For blocks greater than or equal to $\mathcal{B}_\ell$ both statements of Claim 6.8 are trivially satisfied.

First suppose that both statements are true up to some $i-1$, but statement (1) is false for i, i.e. there are distances $e_1, e_2, \ldots, e_k \in S$, $b \in \mathcal{B}_i$ and $e = e_1 \oplus e_2 \oplus \cdots \oplus e_k \in S^\oplus$ with $e \oplus d_i(\ell) \prec \ell$, but $e \oplus b \succeq \ell$. Let (e'_i) be the subsequence of (e_i) containing only distances from blocks larger than $\mathcal{B}_i$ (i.e. blocks $\mathcal{B}_j$ for $j < i$) and (e''_i) be the complement of (e'_i). Let $e' = \bigoplus_i e'_i$ and $e'' = \bigoplus_i e''_i$. Clearly $e = e' \oplus e''$. Finally let $b' = b \oplus e''$.

Now $e' \oplus d_i(\ell) \prec \ell$ and $e' \oplus b' \succeq \ell$. But e' can be expressed as a sum of elements from $S \cap \left(\bigcup_{j<i} \mathcal{B}_j\right)$, hence $e' \in X_{i-1}(\ell)$ and this is a contradiction with the definition of $d_i(\ell)$.

To prove (2), assume that it holds up to $i-1$ and that (1) holds up to i. For a contradiction, suppose that (2) fails for i. This means that there is $e \in \bigcup_{j \leq i} \mathcal{B}_j$ with $e = e_1 \oplus e_2 \oplus \cdots \oplus e_k, e_i \in S \cap \left(\bigcup_{j \leq i} \mathcal{B}_j\right)$ and moreover $e \prec \ell$ and there is $b \in \mathcal{B}_{i+1}$ with $e \oplus b \succeq \ell$ and $e \notin X_i(\ell)$.

As in the previous point, let (e'_i) be the subsequence of (e_i) containing only distances from blocks larger than $\mathcal{B}_i$ (i.e. blocks $\mathcal{B}_j$ for $j < i$) and (e''_i) be the complement of (e'_i). Let $e' = \bigoplus_i e'_i$ and $e'' = \bigoplus_i e''_i$. Clearly $e = e' \oplus e''$,

$e' \prec \ell$ and $e'' \in \mathcal{B}_i$ (from the induction hypothesis it follows that $e'' \neq 0$). But if $e'' \succeq d_i(\ell)$, then already e would be at least ℓ. Hence $e'' \prec d_i(\ell)$ and then from the definition of $X(\mathcal{B}, \ell)$ it follows that $e'' \in X(\mathcal{B}_i, d_i(\ell))$, and thus $e \in X_i(\ell)$ which is a contradiction. This concludes the proof of Claim 6.8. □

The following proposition (an easy consequence of Lemma 6.6) is the main result of this section which will be used in proving the local finiteness property:

Proposition 6.9. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be a distance monoid with finitely many blocks and $S \subset M$ be a finite subset of M. There exists $n = n(S)$ such that for every $\ell \in S$ and every sequence $e_1, e_2, \dots, e_k \in S$ with $\ell \succ e_1 \oplus e_2 \oplus \cdots \oplus e_k$ there is a sequence $f_1, f_2, \dots, f_m \in S$ satisfying the following properties:*

(1) (f_i) is a subsequence of (e_i);
(2) $m < n$; and
(3) if $(f_i) \subsetneq (e_i)$ and a is the largest e_i not in the sequence (f_i), $\mathcal{B}_a$ the block containing a, and $b \in \mathcal{B}_a$ an arbitrary distance, then $\ell \succ b \oplus f_1 \oplus f_2 \oplus \cdots \oplus f_m$.

We will call the distances (f_i) *important.*

Proof. Let $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_b$ be blocks of $\mathfrak{M}$ represented in S by some distance (for each $\mathcal{B}_i$ there is an $a_i \in S$ with $a_i \in \mathcal{B}_i$). For each $\mathcal{B}_i$ define e_i to be the minimum element of $\mathcal{B}_i \cap S$ and let n_i be the smallest integer such that $n_i \times e_i \succeq \mathrm{mus}(\mathcal{B}_i, S)$ (because $\mathcal{B}_i$ is archimedean such n_i exists). Put

$$n = n(S) = 1 + \sum_i n_i.$$

Let $\ell, e_1, e_2, \dots, e_k \in S$ be given with $\ell \succ e_1 \oplus \cdots \oplus e_k$. Now we shall construct the sequence (f_i) satisfying the properties from the statement. For each $\mathcal{B}_i$ create a variable c_i which is initially set to zero. Now go through all e_i and do the following:

(1) Let $\mathcal{B}_j$ be the block containing e_i.
(2) If $c_j \succ \mathrm{mus}(\mathcal{B}_j, S)$, go to the next e_i.
(3) Otherwise put e_i into the (f_i) sequence and increment $c_i := c_i \oplus e_i$.

One can easily check that (f_i) satisfies all properties from the statement. □

6.3. **Convex ordering of $\mathcal{M}_{\mathfrak{M}}$.** To obtain a Ramsey class we need to define a notion of ordering for classes $\mathcal{M}_{\mathfrak{M}}$. The following definition is a generalization of convex ordering for equivalences and metric spaces (see e.g. [16]). Recall that we interpret an $\mathfrak{M}$-metric space as a relational structure $\mathbf{A}$ in the language $L_{\mathfrak{M}}$ with (possibly infinitely many) binary relations R^s, $s \in M^{\succ 0}$.

Definition 6.10. *Given a distance monoid* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *we expand the language* $L_{\mathfrak{M}}$ *to the language* $L^+_{\mathfrak{M}}$ *by a binary relation* $\leq$ *representing a linear order.*

Given an $\mathfrak{M}$*-metric space* $\mathbf{A} = (A, (R^s_{\mathbf{A}})_{s\in M^{\succ 0}})$ *its* convexly ordered lift *is an* $L^+_{\mathfrak{M}}$*-structure* $\mathbf{A}^+ = (A, (R^s_{\mathbf{A}})_{s\in M^{\succ 0}}, \leq_{\mathbf{A}})$ *such that for every block* $\mathcal{B}$ *of* $\mathfrak{M}$ *and every* $a, b, c \in A$ *with* $a \sim_{\mathcal{B}} b$ *and* $a \nsim_{\mathcal{B}} c$ *it holds that* $a \leq_{\mathbf{A}} c$ *if and only if* $b \leq_{\mathbf{A}} c$ *(that is, every ball forms an interval in* $\leq_{\mathbf{A}}$*). We will denote by* $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ *the class of all convexly ordered* $\mathfrak{M}$*-metric spaces.*

We will now consider a further (and more technical) expansion of the class $\mathcal{M}_{\mathfrak{M}}$ which will make it possible to apply Theorem 3.5. 1-1 correspondence between structures in both expansions will let us show Ramsey property of $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$. For this let $B_{\mathfrak{M}}$ be the set of all non-zero non-maximal (in the block order) blocks of $\mathfrak{M}$.

Definition 6.11. *Given a distance monoid* $\mathfrak{M} = (M, \oplus, \preceq, 0)$ *with finitely many blocks denote by* $L^\star_{\mathfrak{M}}$ *the expansion of the language* $L^+_{\mathfrak{M}}$ *adding*

(1) unary functions $F^{\mathcal{B}}$ *for every block* $\mathcal{B} \in B_{\mathfrak{M}}$*, and*
(2) unary functions $F^{\mathcal{B},\mathcal{B}'}$ *for every pair of blocks* $\mathcal{B}, \mathcal{B}' \in B_{\mathfrak{M}}$ *such that* $\mathcal{B} \prec \mathcal{B}'$*.*

For a given convexly ordered metric space $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$*, denote by* $L^\star(\mathbf{A})$ *the* $L^\star$-lift *(or* expansion*) of* $\mathbf{A}$ *created by the following procedure:*

(1) For every $\mathcal{B} \in B_{\mathfrak{M}}$ *enumerate balls of diameter* $\mathcal{B}$ *in* $\mathbf{A}$ *as* $E^1_{\mathcal{B}}, E^2_{\mathcal{B}}, \ldots, E^{n_{\mathcal{B}}}_{\mathcal{B}}$ *in the order of* $\leq_{\mathbf{A}}$ *(recall that balls are linear invervals in* $\leq_{\mathbf{A}}$ *and thus this is well defined).*
(2) For every $\mathcal{B} \in B_{\mathfrak{M}}$ *and* $1 \leq i \leq n_{\mathcal{B}}$ *add a new vertex* $v^i_{\mathcal{B}}$*.*
(3) For every $\mathcal{B} \in B_{\mathfrak{M}}$*,* $1 \leq i \leq n_{\mathcal{B}}$ *and* $v \in E^i_{\mathcal{B}}$ *put* $F^{\mathcal{B}}_{L^\star(\mathbf{A})}(v) = v^i_{\mathcal{B}}$*.*
(4) For every pair of blocks $\mathcal{B}, \mathcal{B}' \in B_{\mathfrak{M}}$ *such that* $\mathcal{B} \prec \mathcal{B}'$ *and every* $1 \leq i \leq n_{\mathcal{B}}$ *put* $F^{\mathcal{B},\mathcal{B}'}_{L^\star(\mathbf{A})}(v^i_{\mathcal{B}}) = F^{\mathcal{B}'}_{L^\star(\mathbf{A})}(v)$ *where* v *is some vertex of* $E^i_{\mathcal{B}}$*.*
(5) Extend linear ordering $\leq_{\mathbf{A}}$ *to ordering* $\leq_{L^{\mathbf{A}}}$ *by putting*
   (a) $v \leq_{L^{\mathbf{A}}} v'$ *for every* $v \in A$*,* $v' \notin A$*,*
   (b) $v^i_{\mathcal{B}} \leq_{L^{\mathbf{A}}} v^j_{\mathcal{B}}$ *for every* $\mathcal{B} \in B_{\mathfrak{M}}$ *and* $1 \leq i \leq j \leq n_{\mathcal{B}}$*, and*
   (c) $v^i_{\mathcal{B}} \leq_{L^{\mathbf{A}}} v^j_{\mathcal{B}'}$ *for every* $\mathcal{B} \prec \mathcal{B}'$*,* $1 \leq i \leq n_{\mathcal{B}}$ *and* $1 \leq j \leq n_{\mathcal{B}'}$*.*
   In $L^\star(\mathbf{A})$ *we will call vertices of* $\mathbf{A}$ *the* original vertices *and the added* $v^i_{\mathcal{B}}$ *vertices the* ball vertices.

Denote by $\mathcal{M}^\star_{\mathfrak{M}}$ *the class of all* $L^\star(\mathbf{A})$*,* $\mathbf{A} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$*.*

Observe that for any structure $\mathbf{A} \in \mathcal{M}^\star_{\mathfrak{M}}$ and any two vertices $u, v \in \mathbf{A}$ we have $F^{\mathcal{B}}_{L^\star(\mathbf{A})}(u) = F^{\mathcal{B}}_{L^\star(\mathbf{A})}(v)$ if and only if $u \sim_{\mathcal{B}} v$.

6.4. **Proof of the main result.** We first prove the main result under the assumption of finitely many blocks.

Lemma 6.12. *Let $\mathfrak{M} = (M, \oplus, \preceq, 0)$ be a distance monoid with finitely many blocks. Then $\mathcal{M}^\star_{\mathfrak{M}}$ is a Ramsey class.*

Proof. We show that $\mathcal{M}^\star_{\mathfrak{M}}$ is an $\mathcal{R}_{\mathfrak{M}}$-multiamalgamation class where the class $\mathcal{R}_{\mathfrak{M}}$ consists of all finite $L^\star_{\mathfrak{M}}$ structures (which is Ramsey by Theorem 3.6).

It remains to verify the locally finite completion property (see Definition 3.4). Let m be the number of blocks of $\mathfrak{M}$. Given $\mathbf{B} \in \mathcal{M}^\star_{\mathfrak{M}}$, let S be the set of all distances in $\mathbf{B}$. Then put

$$n = 2(m+1)n(S),$$

where $n(S)$ is given by Proposition 6.9.

Let $\mathbf{C}'$ be a structure with a homomorphism-embedding to $\mathbf{C}_0 \in \mathcal{R}_{\mathfrak{M}}$ such that every n-element substructure of $\mathbf{C}'$ has a completion in $\mathcal{M}^\star_{\mathfrak{M}}$. Without loss of generality (follows from the definition of completion with respect to copies of $\mathbf{B}$) we may further assume that every vertex, every edge and every pair of vertices with value function defined is contained in some copy of $\mathbf{B}$. We verify that $\mathbf{C}'$ has a strong completion in $\mathcal{M}^\star_{\mathfrak{M}}$.

Vertex set $\mathbf{C}'$ can be split into *original vertices* which correspond to vertices of the underlying metric spaces – for those functions $F^{\mathcal{B}}_{\mathbf{C}'}$ are defined – and *ball vertices* which are values of those functions (and are copies of vertices $v^i_{\mathcal{B}}$ added during the construction of the $L^\star$ lift).

We first show:

Claim 6.13. *For every pair of original vertices $u, v \in \mathbf{C}'$ and block $\mathcal{B} \in B_{\mathfrak{M}}$ such that there exists a walk from u to v where every distance is at most some $a \in \mathcal{B}$ we have $F^{\mathcal{B}}_{\mathbf{C}'}(u) = F^{\mathcal{B}}_{\mathbf{C}'}(v)$.*

Let u' and w' be two neighbouring vertices in the walk. Because every edge of $\mathbf{C}'$ is part of a copy of $\mathbf{B}$ we know that there is a copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}'$ containing both u', w'. It follows that $F^{\mathcal{B}}_{\mathbf{C}'}(u') = F^{\mathcal{B}}_{\mathbf{C}'}(w')$. Consequently all vertices of the walk have same value of $F^{\mathcal{B}}_{\mathbf{C}'}$. This finishes the proof of Claim 6.13.

Denote by $\mathbf{G}$ the $\mathfrak{M}$-graph induced by $\mathbf{C}'$ on the set of original vertices.

Claim 6.14. $\mathbf{G}$ *is $\mathfrak{M}$-metric.*

The main idea of the proof of this claim is to reduce the possibly infinite family of forbidden cycles (given by Proposition 4.3) to a finite family of forbidden substructures by use of the additional ball vertices which serve as shortcuts over unimportant parts of the cycle. An example is sketched in Figure 4.

To the contrary now assume that $\mathbf{G}$ is non-$\mathfrak{M}$-metric. By Proposition 4.3 there exists a non-$\mathfrak{M}$-metric cycle $\mathbf{K}$ in $\mathbf{G}$. We can order the edges in $\mathbf{K}$ as $\ell, e_1, \ldots, e_k$ such that $\ell \succ e_1 \oplus e_2 \oplus \cdots \oplus e_k$. Consider the family (f_i) of important edges in $\mathbf{K}$ given by Proposition 6.9. If $(f_i) = (e_i)$, then the cycle $\mathbf{K}$ has at most $n(S)$ edges, which is a contradiction as every at most n-vertex substructure of $\mathbf{C}'$ has a completion.

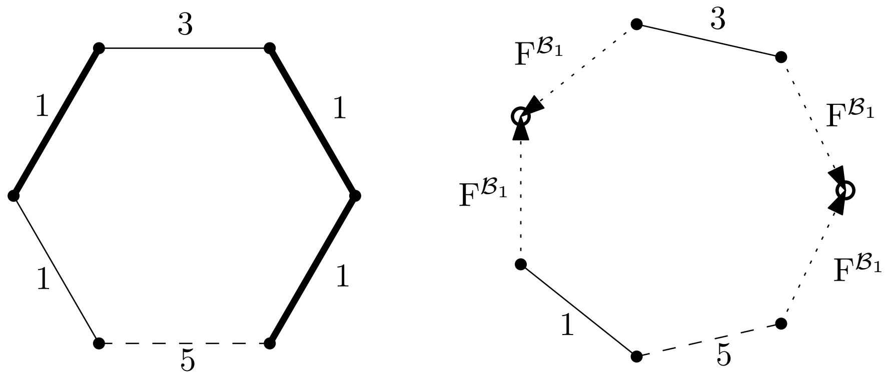

FIGURE 4. Consider monoid $\mathfrak{M}$ given by example 1.9 (1) for $S = \{1, 3, 5\}$. Figure depicts the family of unimportant edges in a non-$\mathfrak{M}$-metric cycle (left), and the corresponding forbidden substructure in $\mathbf{C}'$ (right).

Let $\mathcal{B}$ be the block containing the largest non-important edge as given by Proposition 6.9. Then $\mathcal{B} \in B_{\mathfrak{M}}$ (clearly $\mathcal{B} \neq \{0\}$ and it cannot be the largest block either as all edges from the block surely are important). Hence by Claim 6.13 we know that for every maximal path in $\mathbf{K}$ containing no important edge there exists a vertex $c \in \mathbf{C}'$ such that $F^{\mathcal{B}}_{\mathbf{C}'}(u) = c$ for every vertex u of the path. We call c the *common closure of the path.*

Create $\mathbf{K}'$ as the structure induced by $\mathbf{C}'$ on the set of all vertices of $\mathbf{K}$ which are the endpoints of important edges. Then $\mathbf{K}'$ has at most n vertices ($2n(S)$ accounts for the endpoints of important edges and for each such vertex there are m closure ball vertices) and has no completion, which is a contradiction. This finishes the proof of Claim 6.14.

Claim 6.15. *There exists an $\mathfrak{M}$-metric space $\mathbf{G}'$ which is a strong completion of $\mathbf{G}$ (i.e. $\mathbf{G}$ is $\mathfrak{M}$-metric) and moreover for every $\mathcal{B} \in B_{\mathfrak{M}}$ and every pair u and v of vertices of $\mathbf{G}$, whenever $F^{\mathcal{B}}_{\mathbf{C}'}(u) = F^{\mathcal{B}}_{\mathbf{C}'}(v)$, then $u \sim_{\mathcal{B}} v$ in $\mathbf{G}'$.*

By Claim 6.14 we know that $\mathbf{G}$ is $\mathfrak{M}$-metric and thus there is the shortest path completion of $\mathbf{G}$. This completion however does not necessarily satisfy the moreover part of the statement of Claim 6.15. To satisfy it we first construct $\mathbf{G}_0$ by adding edges to $\mathbf{G}$ such that for every pair of original vertices $u \neq v$ and block $\mathcal{B} \in B_{\mathfrak{M}}$ satisfying $F^{\mathcal{B}}_{\mathbf{C}'}(u) = F^{\mathcal{B}}_{\mathbf{C}'}(v)$ there exists a walk from u to v in $\mathbf{G}_0$ such that every pair of neighbouring vertices is $\sim_{\mathcal{B}}$ equivalent.

New edges to $\mathbf{G}_0$ are added by induction from smallest blocks to largest (ordered by $\preceq$). For every $\mathcal{B}$ follow the following procedure: choose a pair of vertices u and v having common closure for $\mathcal{B}$ (by this we mean that $F^{\mathcal{B}}_{\mathbf{C}'}(u) = F^{\mathcal{B}}_{\mathbf{C}'}(v)$) but there is no walk in $\mathbf{G}'$ such that every pair of neighbouring vertices is $\sim_{\mathcal{B}}$ equivalent) and connect them by an edge of length $\mathrm{mus}(\mathcal{B}, S)$ (c.f. Lemma 6.6). Repeat this procedure as long as such pairs u and v exist.

We shall verify that this process does not introduce a non-$\mathfrak{M}$-metric cycle. Suppose, to the contrary, that $\mathcal{B}$ is the smallest block such that adding an edge of length $\mathrm{mus}(\mathcal{B}, S)$ between vertices u and v created a non-$\mathfrak{M}$-metric cycle $\mathbf{K}$ with edges $\ell, \mathrm{mus}(\mathcal{B}, S), e_1, e_2, \ldots, e_k$ and $\ell \succ \mathrm{mus}(\mathcal{B}, S) \oplus e_1 \oplus e_2 \oplus \cdots \oplus e_k$. Then ℓ must lie in a block larger than $\mathcal{B}$, as the reason for adding the edge was the lack of a $\mathcal{B}$-path between u and v.

Let (e'_i) be the subsequence of e_i containing only edges from blocks larger than $\mathcal{B}$. Then also $\ell \succ \mathrm{mus}(\mathcal{B}, S) \oplus \bigoplus_i e'_i$. And by Lemma 6.6 we know that $\ell \succ b \oplus \bigoplus_i e'_i$ for every $b \in \mathcal{B}$ (thus every $b \in \mathcal{B}'$ where $\mathcal{B}' \preceq \mathcal{B}$). Applying Proposition 6.9 on ℓ and (e'_i), which are all edges in $\mathbf{C}'$ and also lie in the set S, it follows that $\mathbf{C}'$ contained a substructure on at most n vertices with no completion into $\mathcal{M}^\star_{\mathfrak{M}}$, which is a contradiction.

Now $\mathbf{G}'$ can be constructed as the shortest path completion of $\mathbf{G}_0$. This finishes the proof of Claim 6.15

Finally we construct $\mathbf{C}$ from $\mathbf{C}'$ by completing $\mathbf{G}$ to $\mathbf{G}'$ and by completing $\leq_{\mathbf{C}'}$ to a linear order by the following procedure:

Let $\leq_0$ be an arbitrary linear extension of $\leq_{\mathbf{C}'}$. This always exists because of the existence of a homomorphism from $\mathbf{C}'$ to linearly ordered $\mathbf{C}_0$. Enumerate blocks of $\mathfrak{M}$ as $\mathcal{B}_1 \succeq \mathcal{B}_2 \succeq \cdots \succeq \mathcal{B}_p$. Now define $\leq_{\mathbf{C}}$ as follows:

(1) For every pair u, v of original vertices of $\mathbf{C}$ put $u \leq_{\mathbf{C}} v$ if the sequence of vertices $(F^{\mathcal{B}_i}_{\mathbf{C}}(u))_i$ is in order $\leq_0$ lexicographically before $(F^{\mathcal{B}_i}_{\mathbf{C}}(v))_i$ or they are equivalent and $u \leq_0 v$.
(2) For every pair u, v of ball vertices such that u corresponds to block $\mathcal{B}_i$ and v to block $\mathcal{B}_j$ put $u \leq_{\mathbf{C}} v$ if one of the following holds:
  (a) $i < j$,
  (b) the sequence $(F^{\mathcal{B}_i,\mathcal{B}_{i'}}_{\mathbf{C}}(u))_{1\leq i' \leq i}$ is in order $\leq_0$ lexicographically before $(F^{\mathcal{B}_j,\mathcal{B}_{j'}}_{\mathbf{C}}(v))_{1\leq j' \leq j}$.
(3) Finally put $u \leq_{\mathbf{C}} v$ if u is original vertex and v is ball vertex.

$\leq_{\mathbf{C}}$ is clearly a linear oder and by comparing the construction with Definition 6.11 it can be verified that $\leq_{\mathbf{C}}$ and $\leq_{\mathbf{C}'}$ agree on every copy of $\mathbf{B}$ in $\mathbf{C}$ because the convex ordering coincides with the lexicographic ordering according of balls.

It follows that $\mathbf{C}$ is a (strong) completion of $\mathbf{C}'$ in $\overrightarrow{\mathcal{M}}'_{\mathfrak{M}}$. Moreover $\mathbf{C}$ is an $L^\star$ lift of the L^+-structure induced by $\mathbf{C}$ on the original vertices. □

Now we can prove the main theorem of this paper:

Proof of Theorem 1.6. To show that $\overrightarrow{\mathcal{M}}_{\mathfrak{M}}$ is Ramsey consider $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{M}}_{\mathfrak{M}}$. We show that there exists $\mathbf{C} \longrightarrow (\mathbf{B})^{\mathbf{A}}_2$. If $\mathfrak{M}$ has only finitely many blocks, then by Lemma 6.12 there exists $\mathbf{C}' \longrightarrow (L^\star(\mathbf{B}))^{L^\star(\mathbf{A})}_2$. Because $L^\star(\mathbf{B})$ preserves substructures, we can put $\mathbf{C}$ to be the L^+-structure induced by $\mathbf{C}'$ on set of its original vertices.

Now let $\mathfrak{M}$ be an arbitrary distance monoid and let $\mathbf{B} \in \mathcal{M}_{\mathfrak{M}}$. We will observe that $\mathbf{B}$ lies in an amalgamation subclass of $\mathcal{M}_{\mathfrak{M}}$ which happens to be Ramsey. Therefore for every $\mathbf{A}$ which is a substructure of $\mathbf{B}$ and every integer k, we get a $\mathbf{C}$ from the subclass (and hence from $\mathcal{M}_{\mathfrak{M}}$) such that $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_k$, thus proving the Ramsey property for $\mathcal{M}_{\mathfrak{M}}$.

And finding the subclass is actually easy. Let S be the set of all distances which occur in $\mathbf{B}$. Clearly S is finite. Let $\langle S \rangle$ be the submonoid of $\mathfrak{M}$ generated by S (containing every finite sum of elements from S, hence $\langle S \rangle = S^{\oplus} \cup \{0\}$ as defined in Section 6.2). It is easy to check that if we denote by $\oplus'$ and $\preceq'$ the restriction of $\oplus$ and $\preceq$ respectively to $\langle S \rangle$, then $(\langle S \rangle, \oplus', \preceq', 0)$ is a distance monoid with finitely many blocks and hence $\mathbf{B} \in \mathcal{M}_{\langle S \rangle}$. From this point we can proceed by application of Lemma 6.12 as in the first case. □

7. Conclusion

The techniques introduced in this paper can be generalised and used to prove Ramsey property for much wider family of $\mathfrak{M}$-valued metric spaces that contains everything discussed in this paper as well as for example Λ-ultrametric spaces, where Λ is a finite distributive lattice. (These spaces were introduced and their Ramsey expansions were found by Braunfeld [4].) We can also prove EPPA for all those classes (hence extending the results and answering a question of Conant [6]). This can be done by combining the Herwig-Lascar theorem [10] with additional unary functions as done in [9]. Finally, the shortest path completion can be utilised to obtain a Stationary Independence Relation for the $\mathfrak{M}$-valued metric spaces [20]. Our work was also motivated by the analysis of Cherlin's catalogue of metrically homogeneous graphs [5] done in [3, 1, 2] which also gives a rise of monoid-valued metric spaces, however this correspondence is more subtle (the monoids does not satisfy the notion of distance monoid used in this paper).

These results will appear in [12].

References

1. Andres Aranda, David Bradley-Williams, Eng Keat Hng, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and Micheal Pawliuk, *Completing graphs to metric spaces*, Electronic Notes in Discrete Mathematics **61** (2017), 53 – 60, The European Conference on Combinatorics, Graph Theory and Applications (EUROCOMB'17).
2. ———, *Completing graphs to metric spaces*, Submitted, arXiv:1706.00295, 2017.
3. Andres Aranda, David Bradley-Williams, Jan Hubička, Miltiadis Karamanlis, Michael Kompatscher, Matěj Konečný, and Micheal Pawliuk, *Ramsey expansions of metrically homogeneous graphs*, arXiv:1707.02612 (2017), 56p.
4. Samuel Braunfeld, *Ramsey expansions of* Λ*-ultrametric spaces*, preprint, 2017.
5. Gregory Cherlin, *Homogeneous ordered graphs and metrically homogeneous graphs*, In preparation, Jully–August 2016.
6. Gabriel Conant, *Extending partial isometries of generalized metric spaces*, arXiv:1509.04950 (2015).
7. Christian Delhommé, Claude Laflamme, Maurice Pouzet, and Norbert W. Sauer, *Divisibility of countable metric spaces*, European Journal of Combinatorics **28** (2007), no. 6, 1746–1769.

8. Domingos Dellamonica and Vojtěch Rödl, *Distance preserving Ramsey graphs*, Combinatorics, Probability and Computing **21** (2012), no. 04, 554–581.
9. David M. Evans, Jan Hubička, and Jaroslav Nešetřil, *Ramsey properties and extending partial automorphisms for classes of finite structures*, arXiv:1705.02379, 33 pages (2017).
10. Bernhard Herwig and Daniel Lascar, *Extending partial automorphisms and the profinite topology on free groups*, Transactions of the American Mathematical Society **352** (2000), no. 5, 1985–2021.
11. Wilfrid Hodges, *Model theory*, vol. 42, Cambridge University Press, 1993.
12. Jan Hubička, Matěj Konečný, and Jaroslav Nešetřil, *Generalised metric spaces are Ramsey*, in preparation, 2017.
13. Jan Hubička and Jaroslav Nešetřil, *All those Ramsey classes (Ramsey classes with closures and forbidden homomorphisms)*, Submitted, arXiv:1606.07979, 58 pages, 2016.
14. Jaroslav Nešetřil, *Metric spaces are Ramsey*, European Journal of Combinatorics **28** (2007), no. 1, 457–468.
15. Jaroslav Nešetřil and Vojtěch Rödl, *The Ramsey property for graphs with forbidden complete subgraphs*, Journal of Combinatorial Theory, Series B **20** (1976), no. 3, 243–249.
16. Lionel Nguyen Van Thé, *Structural Ramsey theory of metric spaces and topological dynamics of isometry groups*, Memoirs of the American Mathematical Society, American Mathematical Society, 2010.
17. Norbert W. Sauer, *Distance sets of Urysohn metric spaces*, Canadian Journal of Mathematics **65** (2013), no. 1, 222–240.
18. ———, *Oscillation of Urysohn type spaces*, Asymptotic geometric analysis, Springer, 2013, pp. 247–270.
19. Sławomir Solecki, *Extending partial isometries*, Israel Journal of Mathematics **150** (2005), no. 1, 315–331.
20. Katrin Tent and Martin Ziegler, *On the isometry group of the Urysohn space*, Journal of the London Mathematical Society **87** (2013), no. 1, 289–303.
21. Anatoly M. Vershik, *Globalization of the partial isometries of metric spaces and local approximation of the group of isometries of Urysohn space*, Topology and its Applications **155** (2008), no. 14, 1618–1626.
22. Friedrich Wehrung, *Injective positively ordered monoids i*, Journal of Pure and Applied Algebra **83** (1992), no. 1, 43–82.

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