Ramsey properties and extending partial automorphisms for classes of finite structures

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Abstract

We show that every free amalgamation class of finite structures with relations and (symmetric) partial functions is a Ramsey class when enriched by a free linear ordering of vertices. This is a common strengthening of the Nešetřil-Rödl Theorem and the second and third authors' Ramsey theorem for finite models (that is, structures with both relations and functions). We also find subclasses with the ordering property. For languages with relational symbols and unary functions we also show the extension property for partial automorphisms (EPPA) of free amalgamation classes. These general results solve several conjectures and provide an easy Ramseyness test for many classes of structures.

1 Introduction

In this paper we discuss three related concepts — Ramsey classes, the ordering (or lift or expansion) property and the extension property for partial automorphisms (EPPA). The main novelty of our results is that they hold for free amalgamation classes of finite structures with both relations and *partial functions*. This provides a useful tool for proving these types of results for some classes of structures which naturally carry a closure operation. We will explain below what we mean by this; examples and applications are given at the end of the paper.

As is well know, all three of these concepts about classes of finite structures are related to issues in topological dynamics and this relationship provides much of the motivation for what we do. For example, by [20] the automorphism group of the Fraïssé limit of a Ramsey class $\mathcal{R}$ is extremely amenable. Moreover, if the Ramsey class $\mathcal{R}$ has the ordering property with respect to some amalgamation class $\mathcal{K}$, then it determines the universal minimal flow of the automorphism

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group of the Fraïssé limit of $\mathcal{K}$. Thus, our results Theorem 1.3 and Theorem 1.4 about the Ramsey and ordering properties give new examples of this correspondence. By [21] and our Theorem 1.7 about EPPA, the automorphism group of the Fraïssé limit of every free amalgamation class $\mathcal{K}$ in a language where all functions are unary is amenable.

To generalise naturally the known results about relational structures, we need to define carefully what we mean by a structure and substructure, because none of these results holds in the context of free amalgamation classes with strong embeddings as discussed in [10]. In the Ramsey theory setting it is common to work with 'incomplete' structures. This is a problem with the standard model-theoretic notion of structures (see e.g. [14]), where functions are required to be total and thus complete in some sense. Before stating the main results, we give the basic (model-theoretic) setting of this paper. We introduce a variant of the usual model-theoretic structures which allows partial functions and which is well tailored to the Ramsey setting.

Let $L = L_{\mathcal{R}} \cup L_{\mathcal{F}}$ be a language with relational symbols $R \in L_{\mathcal{R}}$ and function symbols $F \in L_{\mathcal{F}}$ each having associated *arities* denoted by $a(R)$ for relations and *domain arity*, $d(F)$, *range arity*, $r(F)$, for functions. Denote by $\binom{A}{n}$ the set of all subsets of A consisting of n elements. An *L-structure* $\mathbf{A}$ is a structure with *vertex set A*, functions $F_{\mathbf{A}} : \mathrm{Dom}(F_{\mathbf{A}}) \to \binom{A}{r(F)}$, $\mathrm{Dom}(F_{\mathbf{A}}) \subseteq A^{d(F)}$, $F \in L_{\mathcal{F}}$ and relations $R_{\mathbf{A}} \subseteq A^{a(R)}$, $R \in L_{\mathcal{R}}$. The set $\mathrm{Dom}(F_{\mathbf{A}})$ is called the *domain* of the function F in $\mathbf{A}$. Notice that the domain is a set of ordered $d(F)$-tuples while the range is set of unordered $r(F)$-tuples. Symmetry in the ranges permits an explicit description of algebraic closures in the Fraïssé limits without changing the automorphism group. It also greatly simplifies some of the notation below.

The language is usually fixed and understood from the context (and it is in most cases denoted by L). If the set A is finite we call $\mathbf{A}$ a *finite structure*. We consider only structures with finitely or countably infinitely many vertices. If the language L contains no function symbols, we call L a *relational language* and say that an L-structure is a *relational L-structure*. A function symbol F such that $d(F) = 1$ is a *unary function.*

The notions of embedding, isomorphism, homomorphisms and free amalgamation classes are natural generalisations of the corresponding notions on relational structures and are formally introduced in Section 1.4. Considering function symbols has important consequences for what we consider a substructure. An L-structure $\mathbf{A}$ is a *substructure* of $\mathbf{B}$ if $A \subseteq B$ and all relations and functions of $\mathbf{B}$ restricted to A are precisely relations and functions of A. In particular if some n-tuple $\vec{t}$ of vertices of A is in $\mathrm{Dom}(F_{\mathbf{B}})$ then it is also in $\mathrm{Dom}(F_{\mathbf{A}})$ and $F_{\mathbf{A}}(\vec{t}) = F_{\mathbf{B}}(\vec{t})$. This implies that $\mathbf{B}$ does not induce a substructure on every subset of $\mathbf{B}$ (but only on 'closed' sets, to be defined later).

Building on these standard model theoretic notions we now outline the contents of this paper. We proceed to the three main directions—Ramsey theory,

the ordering property and the extension property for partial automorphisms. In each of these directions we state the main result. This can be summarised by saying that for free amalgamation classes we have strong positive theorems in each of these areas. There is more to it than just meets the eye: for the first time we demonstrate affinity of all these directions (for the ordering property and Ramsey this is known, but for EPPA much less so).

1.1 Ramsey classes

For structures $\mathbf{A}, \mathbf{B}$ denote by $\binom{\mathbf{B}}{\mathbf{A}}$ the set of all sub-structures of $\mathbf{B}$, which are isomorphic to $\mathbf{A}$. Using this notation the definition of a Ramsey class has the following form. A class $\mathcal{C}$ is a *Ramsey class* if for every two objects $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{C}$ and for every positive integer k there exists a structure $\mathbf{C}$ in $\mathcal{C}$ such that the following holds: For every partition $\binom{\mathbf{C}}{\mathbf{A}}$ into k classes there exists a $\widetilde{\mathbf{B}} \in \binom{\mathbf{C}}{\mathbf{B}}$ such that $\binom{\widetilde{\mathbf{B}}}{\mathbf{A}}$ belongs to one class of the partition. It is usual to shorten the last part of the definition to $\mathbf{C} \longrightarrow (\mathbf{B})_k^{\mathbf{A}}$.

We are motivated by the following, now classical, result.

Theorem 1.1 (Nešetřil-Rödl Theorem [24])**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be ordered hypergraphs. Then there exists an ordered hypergraph $\mathbf{C}$ such that $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$.*

Moreover, if $\mathbf{A}$ and $\mathbf{B}$ do not contain an irreducible hypergraph $\mathbf{F}$ (as an non-induced sub-hypergraph) then $\mathbf{C}$ may be chosen with the same property.

Given a language L, denote by $\overrightarrow{L}$ the language L extended by one binary relation $\leq$. Given an L-structure $\mathbf{A}$, an *ordering of* $\mathbf{A}$ is an $\overrightarrow{L}$-structure extending $\mathbf{A}$ by an arbitrary linear ordering $\leq_{\mathbf{A}}$ of the vertices. For brevity we denote such ordered $\mathbf{A}$ as $\overrightarrow{A}$. Given a class $\mathcal{K}$ of L-structures, denote by $\overrightarrow{\mathcal{K}}$ the class of all orderings of structures in $\mathcal{K}$. We sometimes say that $\overrightarrow{\mathcal{K}}$ arises by taking *free orderings* of structures in $\mathcal{K}$. Theorem 1.1 can now be re-formulated using basic notions of Fraïssé theory (which will be introduced in Section 1.4) as follows:

Theorem 1.2 (Nešetřil-Rödl Theorem for free amalgamation classes)**.** *Let L be a relational language and $\mathcal{K}$ be a free amalgamation class of relational L-structures. Then $\overrightarrow{\mathcal{K}}$ is a Ramsey class.*

The more recent connection between Ramsey classes and extremely amenable groups [20] has motivated a systematic search for new examples. It became apparent that it is important to consider structures with both relations and functions or, equivalently, classes of structures with "strong embeddings". This led to [19] which provides a sufficient structural condition for a subclass of a Ramsey class of structures to be Ramsey and also generalises this approach to classes with formally-described closures. Comparing the two main results of [19] (Theorem 2.1 for classes without closures and Theorem 2.2 for classes with closures) it is clear that considering classes with closures leads to many

technical difficulties. In fact, a recent example given by first author based on Hrushovski's predimension construction [10] not only answers one of main questions in the area (about the existence of precompact Ramsey expansions), but also shows that there is no direct generalisation of the Nešetřil-Rödl Theorem to free amalgamation classes with closures (or strong embeddings). However, perhaps surprisingly, we show that if closures are explicitly represented by means of partial functions, such a statement is true. We prove:

Theorem 1.3. *Let L be a language (involving relational symbols and partial functions) and let $\mathcal{K}$ be a free amalgamation class of L-structures. Then $\overrightarrow{\mathcal{K}}$ is a Ramsey class.*

This yields an alternative proof of the Ramsey property for some recently-discovered Ramsey classes (such as ordered partial Steiner systems [3], bowtie-free graphs [18], bouquet-free graphs [7]) and also for new classes: most importantly a Ramsey expansion of the class of 2-orientations of a Hrushovski predimension construction which is elaborated in [10] and which was one of main motivations for this paper.

1.2 Ordering property

A class $\mathcal{O} \subseteq \overrightarrow{\mathcal{K}}$ has the *ordering property (with respect to $\mathcal{K}$)* if for every $\mathbf{A} \in \mathcal{K}$ there exists $\mathbf{B} \in \mathcal{K}$ such that every ordering $\overrightarrow{B} \in \mathcal{O}$ of $\mathbf{B}$ contains a copy of every ordering $\overrightarrow{A} \in \mathcal{O}$ of $\mathbf{A}$. It is a well known that for every free amalgamation class $\mathcal{K}$ of relational structures the class $\overrightarrow{\mathcal{K}}$ has ordering property. This fact follows by an application of Theorem 1.2 but can also be shown by more general methods based on hypergraphs of large girth [28, 25]. This shows that there are many classes $\mathcal{K}$ of relational structures for which $\overrightarrow{\mathcal{K}}$ has the ordering property (with respect to $\mathcal{K}$) but $\mathcal{K}$ itself is not a Ramsey class.

To see that some extra restriction on our class $\mathcal{O}$ is required, we note the following example. Denote by $\mathcal{T}$ the class of all finite forests of trees represented by a single unary function F connecting a vertex to its father. Let $\mathbf{A}$ be a structure containing two vertices a, b and $F_{\mathbf{A}}(a) = b$. A vertex c is a *root* if $c \notin \mathrm{Dom}(F_{\mathbf{A}})$. Any structure $\mathbf{B}$ can be ordered in increasing order according to the distance from a root vertex. It follows that such an ordering never contains the ordering of $\mathbf{A}$ given by $a \leq_{\mathbf{A}} b$ and consequently $\overrightarrow{\mathcal{T}}$ does not have the ordering property. Nevertheless, we show the following:

Theorem 1.4. *Let L be a language (involving relational symbols and partial functions) and let $\mathcal{K}$ be a free amalgamation class of L-structures. Then there exists an amalgamation class $\mathcal{O} \subseteq \overrightarrow{\mathcal{K}}$ of* admissible orderings *such that:*

  1. every $\mathbf{A} \in \mathcal{K}$ has an ordering in $\mathcal{O}$;

  2. $\mathcal{O}$ is a Ramsey class; and,

  3. $\mathcal{O}$ has the ordering property (with respect to $\mathcal{K}$).

The details of the admissible orderings are technical and are described in full in Definition 3.2. The proof of Theorem 1.4 is a combination of the Ramsey methods used to show the ordering property of classes of relational structures and the methods used to show the ordering property of classes with unary functions (elaborated in [10]).

1.3 Extension property for partial automorphisms – EPPA

A *partial automorphism* of an L-structure $\mathbf{A}$ is an isomorphism $f : \mathbf{D} \to \mathbf{E}$ for some $\mathbf{D}, \mathbf{E}$ being substructures of $\mathbf{A}$. We say that a class of finite L-structures $\mathcal{K}$ has the *extension property for partial automorphisms* (*EPPA*, sometimes called the *Hrushovski extension property*) if whenever $\mathbf{A} \in \mathcal{K}$ there is $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}$ is substructure of $\mathbf{B}$ and every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$, see [17, 12, 13, 16, 33, 35]. In the following we will simply call $\mathbf{B}$ with the property above an *EPPA-extension* of $\mathbf{A}$.

For relational languages, the extension property for partial automorphisms of free amalgamation classes can be either derived from the Herwig-Lascar Theorem [13] or in a more combinatorial way from the following strengthening of the extension property for partial automorphisms:

Theorem 1.5 (Hodkinson-Otto [16])**.** *Let L be a relational language, then for every finite L-structure $\mathbf{A}$ there exists a finite clique faithful EPPA-extension $\mathbf{B}$.*

A *clique faithful EPPA-extension* $\mathbf{B}$ is an EPPA-extension of $\mathbf{A}$ with the additional property that for every clique $\mathbf{C}$ in the Gaifman graph of $\mathbf{B}$ there exists an automorphism g of $\mathbf{B}$ such that $g(C) \subseteq A$. It is a well known fact that free amalgamation classes can be equivalently described by forbidden embeddings from a family of structures whose Gaifman graph is a clique and consequently Theorem 1.5 implies that every free amalgamation class of relational structures has EPPA.

The notion of *irreducibility* of a structure (given in Definition 2.1) is a natural generalisation to the context of functional languages of the above notion of a clique in a graph. We say that an EPPA-extension $\mathbf{B}$ of $\mathbf{A}$ is *irreducible substructure faithful* if for every irreducible substructure $\mathbf{C}$ of $\mathbf{B}$ there exists an automorphism g of $\mathbf{B}$ such that $g(C) \subseteq A$.

Theorem 1.5 was further strengthened by Siniora and Solecki in the following form.

Theorem 1.6 (Siniora-Solecki [32])**.** *Let L be relational language. Then for every finite relational L-structure $\mathbf{A}$ there exists a finite clique faithful and coherent EPPA-extension $\mathbf{B}$.*

Let X be a set and $\mathcal{P}$ be a family of partial bijections between subsets of X. A triple (f, g, h) from $\mathcal{P}$ is called a *coherent triple* if $\mathrm{Dom}(f) = \mathrm{Dom}(h), \mathrm{Range}(f) = \mathrm{Dom}(g), \mathrm{Range}(g) = \mathrm{Range}(h)$ and $h = g \circ f$.

Let X and Y be sets, and $\mathcal{P}$ and $\mathcal{Q}$ be families of partial bijections between subsets of X and between subsets of Y, respectively. A function $\varphi : \mathcal{P} \to \mathcal{Q}$ is said to be a *coherent map* if for each coherent triple (f, g, h) from $\mathcal{P}$, its image $\varphi(f), \varphi(g), \varphi(h)$ in $\mathcal{Q}$ is coherent.

An EPPA-extension **B** of **A** is *coherent* if every partial automorphism f extends to some $\hat{f} \in \operatorname{Aut}(\mathbf{B})$ with the property that the map φ from partial automorphisms of **A** to automorphisms of **B** given by $\varphi(f) = \hat{f}$ is coherent.

Our third main result is a strengthening of the above results to classes of structures with unary functions. The unarity is important in our construction.

Theorem 1.7. *Let L be language such that every function $F \in L$ is unary. Then for every finite L-structure **A** there exists a finite, irreducible substructure faithful, coherent EPPA-extension **B**. Consequently every free amalgamation class $\mathcal{K}$ of finite L-structures has the coherent extension property for partial automorphisms.*

1.4 Further background and notation

We now review some standard graph-theoretic and model-theoretic notions (see e.g. [14]).

A *homomorphism* $f : \mathbf{A} \to \mathbf{B}$ is a mapping $f : A \to B$ such that for every $R \in L_{\mathcal{R}}$ and $F \in L_{\mathcal{F}}$ we have:

(a) $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \implies (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$, and,

(b) $f(\operatorname{Dom}(F_{\mathbf{A}})) \subseteq \operatorname{Dom}(F_{\mathbf{B}})$ and

$$f(F_{\mathbf{A}}(x_1, c_2, \ldots, x_{d(F)})) = F_{\mathbf{B}}(f(x_1), f(x_2), \ldots, f(x_{d(F)})).$$

For a subset $A' \subseteq A$ we denote by $f(A')$ the set $\{f(x) : x \in A'\}$ and by $f(\mathbf{A})$ the homomorphic image of a structure **A**.

If f is injective, then f is called a *monomorphism*. A monomorphism f is an *embedding* if for every $R \in L_{\mathcal{R}}$ and $F \in L_{\mathcal{F}}$:

(a) $(x_1, x_2, \ldots, x_{a(R)}) \in R_{\mathbf{A}} \iff (f(x_1), f(x_2), \ldots, f(x_{a(R)})) \in R_{\mathbf{B}}$, and,

(b) for every $F \in L_{\mathcal{F}}$ it holds that

$$(x_1, \ldots, x_{d(F)}) \in \operatorname{Dom}(F_{\mathbf{A}}) \iff (f(x_1), \ldots, f(x_{d(F)})) \in \operatorname{Dom}(F_{\mathbf{B}}).$$

If f is an embedding which is an inclusion then **A** is a *substructure* (or *subobject*) of **B**. For an embedding $f : \mathbf{A} \to \mathbf{B}$ we say that **A** is *isomorphic* to $f(\mathbf{A})$ and $f(\mathbf{A})$ is also called a *copy* of **A** in **B**. Thus $\binom{\mathbf{B}}{\mathbf{A}}$ is defined as the set of all copies of **A** in **B**.

Given $\mathbf{A} \in \mathcal{K}$ and $B \subset A$, the *closure of B in **A***, denoted by $\operatorname{Cl}_{\mathbf{A}}(B)$, is the smallest substructure of **A** containing B. Closure in **A** is *unary* if $\operatorname{Cl}_{\mathbf{A}}(B) = \bigcup_{v \in B} \operatorname{Cl}_{\mathbf{A}}(v)$ for all $B \subset A$.
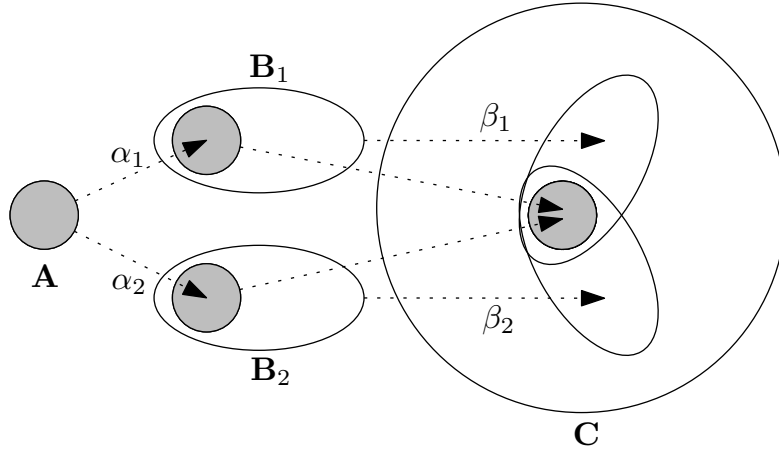
Figure 1: An amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$.

Let $\mathbf{A}$, $\mathbf{B}_1$ and $\mathbf{B}_2$ be structures, α_1 an embedding of $\mathbf{A}$ into $\mathbf{B}_1$ and α_2 an embedding of $\mathbf{A}$ into $\mathbf{B}_2$. Then every structure $\mathbf{C}$ with embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}$ and $\beta_2 : \mathbf{B}_2 \to \mathbf{C}$ such that $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is called an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2. See Figure 1. We will call $\mathbf{C}$ simply an *amalgamation* of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ (as in most cases α_1 and α_2 can be chosen to be inclusion embeddings).

We say that the amalgamation is *free* if $\beta_1(x_1) = \beta_2(x_2)$ if and only if $x_1 \in \alpha_1(A)$ and $x_2 \in \alpha_2(A)$ and there are no tuples in any relations of $\mathbf{C}$ and no tuples in $\mathrm{Dom}(F_{\mathbf{C}})$, $F \in L_{\mathcal{F}}$, using vertices of both $\beta_1(B_1 \setminus \alpha_1(A))$ and $\beta_2(B_2 \setminus \alpha_2(A))$.

Definition 1.1. An *amalgamation class* is a class $\mathcal{K}$ of finite structures satisfying the following three conditions:

1. *Hereditary property:* For every $\mathbf{A} \in \mathcal{K}$ and a substructure $\mathbf{B}$ of $\mathbf{A}$ we have $\mathbf{B} \in \mathcal{K}$;

2. *Joint embedding property:* For every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ there exists $\mathbf{C} \in \mathcal{K}$ such that $\mathbf{C}$ contains both $\mathbf{A}$ and $\mathbf{B}$ as substructures;

3. *Amalgamation property:* For $\mathbf{A}, \mathbf{B}_1, \mathbf{B}_2 \in \mathcal{K}$ and α_1 embedding of $\mathbf{A}$ into $\mathbf{B}_1$, α_2 embedding of $\mathbf{A}$ into $\mathbf{B}_2$, there is $\mathbf{C} \in \mathcal{K}$ which is an amalgamation of $\mathbf{B}_1$ and $\mathbf{B}_2$ over $\mathbf{A}$ with respect to α_1 and α_2.

If the $\mathbf{C}$ in the amalgamation property can always be chosen as the free amalgamation, then $\mathcal{K}$ is a *free amalgamation class*.

We will give examples of free amalgamation classes (in the case where the language L is not relational) in Section 5.

2 Free amalgamation classes are Ramsey

The proof of Theorem 1.3 is a variation of the Partite Construction introduced by Nešetřil and Rödl for classes of hypergraphs and relational structures

(see [27]) which was recently extended to classes with unary [18] and later general closures [19, 3]. The Partite Construction is machinery which allows one to transform one Ramsey class into another, more special, Ramsey class. What follows is an induced partite construction proof (as used previously, for example, in [23]) done in the context of structures involving partial functions.

The basic Ramsey class in our construction will be provided by the following result about the Ramsey property of ordered structures with relations and (total) functions.

Theorem 2.1 (Ramsey theorem for finite models, Theorem 4.26 of [19]). *Suppose L is a language containing a relational symbol $\leq$ and let $\overrightarrow{\mathrm{Str}}(L)$ be the class of all finite L-structures where $\leq$ is a linear ordering of the vertices (and all functions are total). Then $\overrightarrow{\mathrm{Str}}(L)$ is a Ramsey class.*

This is a strengthening of the theorem giving the Ramsey property of ordered relational structures proved independently by Nešetřil-Rödl [29] and Abramson-Harrington [1] in 1970s. Considering functions is a rather difficult task and the proof of Theorem 2.1 involves a recursive nesting of the Partite Constructions to establish valid non-unary closures. Building on Theorem 2.1 our task is significantly easier and we only concentrate on further refining the Ramsey structure given by Theorem 2.1 into one belonging to a given free amalgamation class. This is done by tracking all irreducible substructures of the object constructed.

An irreducible hypergraph, as considered in Theorem 1.1, is a hypergraph where every pair of distinct vertices is covered by a hyper-edge (this concept was introduced as a direct generalisation of complete graphs). Analogously, a relational structure $\mathcal{F}$ is *irreducible* if every pair of vertices belongs to some tuple in a relation of $\mathcal{F}$. It is well known that every free amalgamation class $\mathcal{K}$ of relational structures can be equivalently described as a class of finite structures that contains no copies of structures from a fixed family $\mathcal{F}$ of irreducible relational structures. In fact the family $\mathcal{F}$ consists of all minimal structures not belonging to class $\mathcal{K}$. This easy observation explains the correspondence of Theorem 1.1 and Theorem 1.2.

Our construction is based on the following refinement of the notion of irreducible structure in the context of structures with partial functions (which allows us to strengthen the construction in [19]):

Definition 2.1. An L-structure $\mathbf{A}$ is *irreducible* if it cannot be created as a free amalgamation of two of its proper substructures.

Example. Consider the language L consisting of one binary relation R and one unary function F. An example of an irreducible structure is a structure $\mathbf{A}$ (depicted in Figure 2) on vertices $A = \{a, b, c, d\}$ where $(a, b) \in R_{\mathbf{A}}$, $\mathrm{Dom}(F_{\mathbf{A}}) = \{a, b\}$ and $F_{\mathbf{A}}(a) = \{c\}$, $F_{\mathbf{A}}(b) = \{d\}$. This structure is reducible if F is seen as a relation rather than function.
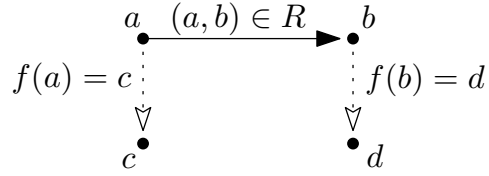
Figure 2: An example of an irreducible structure with a binary relation R and a unary function F.

The basic part of our construction of Ramsey objects is a variant of the Partite Lemma (introduced in [27] and refined for closures in [19]) which deals with the following objects.

Definition 2.2 (**A**-partite system)**.** Let L be a language and $\mathbf{A}$ an L-structure. Assume $A = \{1, 2, \ldots, a\}$. An $\mathbf{A}$-*partite L-system* is a tuple $(\mathbf{A}, \mathcal{X}_{\mathbf{B}}, \mathbf{B})$ where $\mathbf{B}$ is an L-structure and $\mathcal{X}_{\mathbf{B}} = \{X_{\mathbf{B}}^1, X_{\mathbf{B}}^2, \ldots, X_{\mathbf{B}}^a\}$ is a partition of the vertex set of $\mathbf{B}$ into a classes $X_{\mathbf{B}}^i$, called *parts* of $\mathbf{B}$, such that

1. the mapping π which maps every $x \in X_{\mathbf{B}}^i$ to i, $i = 1, 2, \ldots, a$, is a homomorphism $\mathbf{B} \to \mathbf{A}$ (π is called the *projection*);

2. every tuple in every relation of $\mathbf{B}$ meets every class $X_{\mathbf{B}}^i$ in at most one element (i.e. these tuples are called *transversal* with respect to the partition).

3. for every function $F \in L_{\mathcal{F}}$ it holds that for every $\vec{t} \in \mathrm{Dom}(F_{\mathbf{B}})$ the tuple created by concatenation of $\vec{t}$ and $F_{\mathbf{B}}(\vec{t})$ (in any order) is transversal.

The *isomorphisms* and *embeddings* of $\mathbf{A}$-partite systems, say of $\mathbf{B}_1$ into $\mathbf{B}_2$, are defined as the isomorphisms and embeddings of structures together with the condition that all parts are being preserved (the part $X_{\mathbf{B}_1}^i$ is mapped to $X_{\mathbf{B}_2}^i$ for every $i = 1, 2, \ldots, a$). Of course, $\mathbf{A}$ itself can be considered as an $\mathbf{A}$-partite system.

We say that an $\mathbf{A}$-partite L-system is *transversal* if all of its parts consist of at most one vertex.

Lemma 2.2 (Partite Lemma with relations and functions)**.** *Let L be a language, $\mathbf{A}$ be a finite L-structure, and $\mathbf{B}$ be a finite $\mathbf{A}$-partite L-system. Then there exists a finite $\mathbf{A}$-partite L-system $\mathbf{C}$ such that*

$$\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

Moreover if every irreducible subsystem of $\mathbf{B}$ is transversal, then we can also ensure that every irreducible subsystem of $\mathbf{C}$ is transversal.

(Compare the Partite Lemma in [19]. Here we newly introduced the statement about transversality of irreducible substructures. Note that the embeddings considered in the Ramsey statement are all as $\mathbf{A}$-partite systems.)

For completeness, we briefly recall the Hales-Jewett Theorem [11]. Consider the family of functions $f : \{1, 2, \ldots, N\} \to \Sigma$ for some finite alphabet Σ. A *combinatorial line* $\mathcal{L}$ is a pair (ω, h) where $\emptyset \neq \omega \subseteq \{1, 2, \ldots, N\}$ and h is a function from $\{1, 2, \ldots, N\} \setminus \omega$ to Σ. The combinatorial line $\mathcal{L}$ describes the family of all those functions $f : \{1, 2, \ldots, N\} \to \Sigma$ that are constant on ω and $f(i) = h(i)$ otherwise. The Hales-Jewett Theorem guarantees, for sufficiently large N, that for every 2-colouring of the functions $f : \{1, 2, \ldots, N\} \to \Sigma$ there exists a monochromatic combinatorial line.

Proof of Lemma 2.2. Assume without loss of generality $A = \{1, 2, \ldots, a\}$ and denote by $\mathcal{X}_{\mathbf{B}} = \{X_{\mathbf{B}}^1, X_{\mathbf{B}}^2, \ldots, X_{\mathbf{B}}^a\}$ the parts of $\mathbf{B}$. We take N sufficiently large (that will be specified later) and construct an $\mathbf{A}$-partite L-system $\mathbf{C}$ with parts $\mathcal{X}_{\mathbf{C}} = \{X_{\mathbf{C}}^1, X_{\mathbf{C}}^2, \ldots, X_{\mathbf{C}}^a\}$ as follows.

1. For every $1 \leq i \leq a$ let $X_{\mathbf{C}}^i$ be the set of all functions

$$f : \{1, 2, \ldots, N\} \to X_{\mathbf{B}}^i.$$

2. For every relation $R \in L_{\mathcal{R}}$, put

$$(f_1, f_2, \ldots, f_{a(R)}) \in R_{\mathbf{C}}$$

if and only if for every $1 \leq i \leq N$ it holds that

$$(f_1(i), f_2(i), \ldots, f_{a(R)}(i)) \in R_{\mathbf{B}}.$$

3. For every function $F \in L_{\mathcal{F}}$, put

$$F_{\mathbf{C}}(f_1, f_2 \ldots, f_{d(F)})(i) = F_{\mathbf{B}}(f_1(i), f_2(i), \ldots, f_{d(F)}(i))$$

if and only if for every $j = 1, 2, \ldots, N$

$$(f_1(j), f_2(j), \ldots, f_{d(F)}(j)) \in \mathrm{Dom}(F_{\mathbf{B}}).$$

This completes the (product) construction of $\mathbf{C}$. It is easy to check that $\mathbf{C}$ is indeed an $\mathbf{A}$-partite L-system with parts $\mathcal{X}_{\mathbf{C}} = \{X_{\mathbf{C}}^1, X_{\mathbf{C}}^2, \ldots, X_{\mathbf{C}}^a\}$.

We verify that, if N is large enough, $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$. Let $\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_t$ be an enumeration of all subsystems of $\mathbf{B}$ which are isomorphic to $\mathbf{A}$. Put $\Sigma = \{1, 2, \ldots, t\}$ which we consider as an alphabet. Each combinatorial line $\mathcal{L} = (\omega, h)$ in Σ^N corresponds to an embedding $e_{\mathcal{L}} : \mathbf{B} \to \mathbf{C}$ which assigns to every vertex $v \in X_{\mathbf{B}}^p$ a function $e_{\mathcal{L}}(v) : \{1, 2, \ldots, N\} \to X_{\mathbf{B}}^p$ (i.e. a vertex of $X_{\mathbf{C}}^p$) such that:

$$e_{\mathcal{L}}(v)(i) = \begin{cases} v \text{ for } i \in \omega, \text{ and,} \\ \text{the unique vertex in } \widetilde{A}_{h(i)} \cap X_{\mathbf{B}}^p \text{ otherwise.} \end{cases}$$

It follows from the construction of $\mathbf{C}$ and from the fact that $\mathbf{B}$ has a projection to $\mathbf{A}$ that $e_{\mathcal{L}}$ is an embedding.

Let N be the Hales-Jewett number guaranteeing a monochromatic line in any 2-colouring of the N-dimensional cube over an alphabet Σ. Now assume that $\mathcal{A}_1, \mathcal{A}_2$ is a 2-colouring of all copies of $\mathbf{A}$ in $\mathbf{C}$. Using the construction of $\mathbf{C}$ we see that among copies of $\mathbf{A}$ are copies induced by an N-tuple $(\widetilde{\mathbf{A}}_{u(1)}, \widetilde{\mathbf{A}}_{u(2)}, \ldots, \widetilde{\mathbf{A}}_{u(N)})$ of copies of $\mathbf{A}$ in $\mathbf{B}$ for every function $u : \{1, 2, \ldots, N\} \to \{1, 2, \ldots, t\}$. However such copies are coded by the elements of the cube $\{1, 2, \ldots, t\}^N$ and thus there is a monochromatic combinatorial line $\mathcal{L}$. The monochromatic copy of $\mathbf{B}$ is then $e_{\mathcal{L}}(\mathbf{B})$.

Finally we verify that if every irreducible subsystem of $\mathbf{B}$ is transversal then also every irreducible subsytem of $\mathbf{C}$ is transversal. Assume the contrary and denote by $\mathbf{D}$ a non-transversal irreducible subsystem of $\mathbf{C}$. Denote by $f_1, f_2, \ldots, f_n$ an enumeration of all distinct vertices of $\mathbf{D}$. By non-transversality assume that f_1 and f_2 are in the same part. For every $i \in 1, 2, \ldots, N$ denote by $\mathbf{D}_i$ the substructure of $\mathbf{B}$ on vertices $f_1(i), f_2(i), \ldots, f_n(i)$. Because $\mathbf{D}_i$ is a homomorphic image of $\mathbf{D}$ it follows that $\mathbf{D}_i$ is transversal. Consequently $f_1(i) = f_2(i)$. Because this holds for every choice of i, we have $f_1 = f_2$. A contradiction. $\qquad\square$

Proof of Theorem 1.3. Given $\mathbf{A}, \mathbf{B} \in \overrightarrow{\mathcal{K}} \subseteq \overrightarrow{\mathrm{Str}}(\overrightarrow{L})$ use Theorem 2.1 to obtain

$$\mathbf{C}_0 \longrightarrow (\mathbf{B})_2^{\mathbf{A}}.$$

This is clearly possible when all functions in $\mathbf{B}$ are total. In case they are not, it is possible to extend the language L by new relations that represent the domain of each partial function and turn every symmetric partial function $F \in L_{\mathcal{F}}$ to a complete function by defining $F_{\mathbf{B}}(\vec{t}) = \vec{u}$, where $\vec{u}$ is a tuple consisting of the minimal vertex of $\vec{t}$ (in the order $\leq_{\mathbf{B}}$) repeated $r(F)$ times, and by ordering tuples using $\leq$. Such a 'completion' makes every function total and preserve all copies of $\mathbf{A}$ and $\mathbf{B}$. Once Theorem 2.1 has been applied, we pass back to the original language and remove what was added in the 'completion'.

Enumerate all copies of $\mathbf{A}$ in $\mathbf{C}_0$ as $\{\widetilde{\mathbf{A}}_1, \widetilde{\mathbf{A}}_2, \ldots, \widetilde{\mathbf{A}}_b\}$. We will define $\mathbf{C}_0$-partite systems ('pictures') $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_b$ such that

(i) every irreducible subsystem $\mathbf{E}$ of $\mathbf{P}_k$ is transversal and (if seen as a structure) is isomorphic to some substructure of $\mathbf{B}$, and,

(ii) in any 2-colouring of $\binom{\mathbf{P}_k}{\mathbf{A}}$, $1 \leq k \leq b$, there exists a copy $\widetilde{\mathbf{P}}_{k-1}$ of $\mathbf{P}_{k-1}$ such that all copies of $\mathbf{A}$ with a projection to $\widetilde{\mathbf{A}}_k$ are monochromatic.

We then show that putting $\mathbf{C}$ to be $\mathbf{P}_b$ (seen as a structure) with the linear order completed arbitrarily, we have the desired Ramsey property $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$.

We first verify that if (i) holds, then $\mathbf{C} \in \overrightarrow{\mathcal{K}}$. Assume the contrary. Denote by $\mathbf{F}$ the minimal substructure of $\mathbf{C}$ such that $\mathbf{F} \notin \overrightarrow{\mathcal{K}}$. Because $\mathcal{K}$ is a free
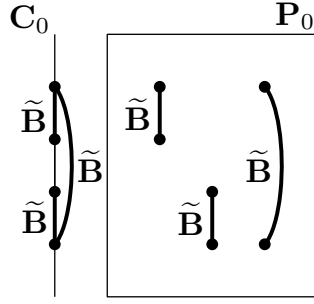
Figure 3: The construction of $\mathbf{P}_0$.

amalgamation class, we know that the unordered shadow of $\mathbf{F}$ is irreducible. By (i) however $\mathbf{F}$ is a substructure of $\mathbf{B} \in \mathcal{K}$. A contradiction to $\mathcal{K}$ being hereditary.

It remains to prove (i) and (ii). Put $C_0 = \{1, \ldots, c\}$ and $\mathcal{X}_{\mathbf{P}_k} = \{X_k^1, X_k^2, \ldots, X_k^c\}$. We proceed by induction on k.

1. The picture $\mathbf{P}_0$ is constructed as a disjoint union of copies of $\mathbf{B}$. For every copy $\widetilde{\mathbf{B}}$ of $\mathbf{B}$ in $\mathbf{C}_0$ we include a copy $\widetilde{\mathbf{B}}'$ of $\widetilde{\mathbf{B}}$ in $\mathbf{P}_0$ which projects onto $\widetilde{\mathbf{B}}$. The copies corresponding to different $\widetilde{\mathbf{B}}$ are disjoint (see Figure 3). This indeed satisfies (i).

2. Suppose the picture $\mathbf{P}_k$ is already constructed. Let $\mathbf{B}_k$ be the substructure of $\mathbf{P}_k$ induced by $\mathbf{P}_k$ on vertices which project to $\widetilde{\mathbf{A}}_{k+1}$. We use the Partite Lemma 2.2 to obtain an $\widetilde{\mathbf{A}}_{k+1}$-partite system $\mathbf{D}_{k+1}$ with $\mathbf{D}_{k+1} \longrightarrow (\mathbf{B}_k)_2^{\widetilde{\mathbf{A}}_{k+1}}$. Now consider all copies of $\mathbf{B}_k$ in $\mathbf{D}_{k+1}$ and extend each of these structures to a copy of $\mathbf{P}_k$ (using free amalgamation of $\mathbf{C}_0$-partite systems). These copies are disjoint outside $\mathbf{D}_{k+1}$. In this extension we preserve the parts of all the copies. The result of this multiple amalgamation is $\mathbf{P}_{k+1}$. Because $\mathbf{D}_{k+1} \longrightarrow (\mathbf{B}_k)_2^{\widetilde{\mathbf{A}}_{k+1}}$ we know that $\mathbf{P}_{k+1}$ satisfies (ii).

   From the 'Moreover' part of Lemma 2.2, we may assume that $\mathbf{D}_{k+1}$ satisfies (i). Because $\mathbf{P}_{k+1}$ is created by a series of free amalgamations, it follows that $\mathbf{P}_{k+1}$ also satisfies (i).

Put $\mathbf{C} = \mathbf{P}_b$. It follows easily that $\mathbf{C} \longrightarrow (\mathbf{B})_2^{\mathbf{A}}$. Indeed, by a backward induction on k one proves that in any 2-colouring of $\binom{\mathbf{C}}{\mathbf{A}}$ there exists a copy $\widetilde{\mathbf{P}}_0$ of $\mathbf{P}_0$ such that the colour of a copy of $\mathbf{A}$ in $\widetilde{\mathbf{P}}_0$ depends only on its projection to $\mathbf{C}_0$. As this in turn induces a colouring of the copies of $\mathbf{A}$ in $\mathbf{C}_0$, we obtain a monochromatic copy of $\mathbf{B}$ in $\widetilde{\mathbf{P}}_0$. $\qquad \square$

Remark. This proof of Theorem 1.3 follows the Partite Construction developed in [26, 27]. It may be seen as a cleaner proof of a particular case of the more general (and more elaborate) Theorem 2.2 of [19].

3 Ordering property

For many Ramsey classes $\mathcal{K}$, the class $\overrightarrow{\mathcal{K}}$ of free orderings of structures in $\mathcal{K}$ has the ordering property. In fact, for free amalgamation classes we can give a full characterisation by means of the following easy (and folklore) proposition.

Proposition 3.1. *Let $\mathcal{K}$ be a free amalgamation class of L-structures. Then $\overrightarrow{\mathcal{K}}$ has the ordering property if and only if for all $u, v \in \mathbf{A} \in \mathcal{K}$ it holds that both $\mathrm{Cl}_\mathbf{A}(u)$ and $\mathrm{Cl}_\mathbf{A}(v)$ are trivial (consist of a single vertex) and are isomorphic.*

Remark. It does not follow from Proposition 3.1 that the class $\mathcal{K}$ must have only trivial closures in order for $\overrightarrow{\mathcal{K}}$ to have the ordering property. Consider for example a class of structures with one binary function F such that the domain of F does not contain tuples with duplicated vertices. Here all vertices are closed, but pairs of vertices have non-trivial closures. A related example is discussed in Section 5.2.

Proof. Assume that there is $\overrightarrow{\mathbf{A}} \in \mathcal{K}$ and $u, v \in \overrightarrow{\mathbf{A}}$ such that $\mathrm{Cl}_\mathbf{A}(u)$ is not isomorphic to $\mathrm{Cl}_\mathbf{A}(v)$. Then one can choose an ordering $\overrightarrow{\mathbf{A}}$ of $\mathbf{A}$ so that the set of all vertices $v' \in A$ such that $\mathrm{Cl}_\mathbf{A}(v')$ is isomorphic to $\mathrm{Cl}_\mathbf{A}(v)$ forms an initial segment. Now assume, to the contrary, that there exists $\mathbf{B} \in \mathcal{K}$ such that every ordering of $\mathbf{B}$ contains a copy of $\overrightarrow{\mathbf{A}}$. This is clearly not possible because one can choose an ordering of $\mathbf{B}$ such that all vertices $u' \in B$ such that $\mathrm{Cl}_\mathbf{A}(u')$ is isomorphic to $\mathrm{Cl}_\mathbf{A}(u)$ forms an initial segment. This is a contradiction to $\overrightarrow{\mathcal{K}}$ having the ordering property.

Now consider the case that all closures of vertices are isomorphic, but not trivial. Because the intersection of two closures is also closed, it follows that they are disjoint and thus it is always possible to order structures in a way that all vertex closures form intervals. It follows that there is no $\mathbf{B}$ witnessing the ordering property for any $\overrightarrow{\mathbf{A}} \in \mathcal{K}$ which is ordered so that some vertex closure is not an interval.

Finally assume that $\mathcal{K}$ is a free amalgamation class where all closures of vertices are trivial and mutually isomorphic. We may assume that there are no unary relations. Given $\overrightarrow{\mathbf{A}} \in \overrightarrow{\mathcal{K}}$ we construct $\overrightarrow{\mathbf{B}}_0$ from a disjoint copy of $\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}$ (by this we mean a structure created from $\overrightarrow{\mathbf{A}}$ by reversing the linear ordering of vertices). Now extend $\overrightarrow{\mathbf{B}}_0$ to $\overrightarrow{\mathbf{B}}_1$ by adding, between every neighbouring pair of vertices $u \leq_{\overrightarrow{\mathbf{B}}_0} v \in B_0$ a new vertex $n_{u,v}$. Extend the order so $u \leq_{\overrightarrow{\mathbf{B}}_1} n_{u,v} \leq_{\overrightarrow{\mathbf{B}}_1} v$. By free amalgamation, $\overrightarrow{\mathbf{B}}_1 \in \overrightarrow{\mathcal{K}}$.

Denote by $\overrightarrow{\mathbf{I}} \in \overrightarrow{\mathcal{K}}$ a structure consisting of two vertices $u \leq_{\overrightarrow{\mathbf{I}}} v$ and no relations containing both of them besides $\leq_{\overrightarrow{\mathbf{I}}}$. Find $\overrightarrow{\mathbf{B}} \in \overrightarrow{\mathcal{K}}$ with $\overrightarrow{\mathbf{B}} \longrightarrow (\overrightarrow{\mathbf{B}}_1)_2^{\overrightarrow{\mathbf{I}}}$ by application of Theorem 1.3.

We verify that $\overrightarrow{\mathbf{B}}$ has the desired property. Let $\overrightarrow{\mathbf{C}}$ be any re-ordering of $\overrightarrow{\mathbf{B}}$. This re-ordering induces a coloring of $\binom{\overrightarrow{\mathbf{B}}}{\overrightarrow{\mathbf{I}}}$: if the order of the points in some $\overrightarrow{\mathbf{I}}' \in \binom{\overrightarrow{\mathbf{B}}}{\overrightarrow{\mathbf{I}}}$ agrees with thier order in $\overrightarrow{\mathbf{C}}$, then color $\overrightarrow{\mathbf{I}}'$ red, and blue otherwise.
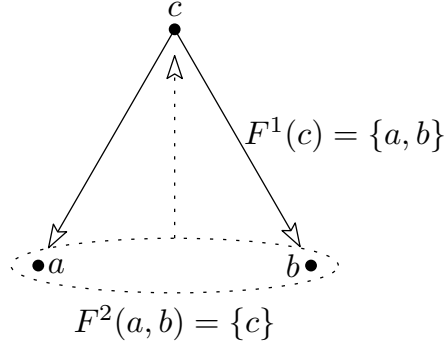
$$F^1(c) = \{a, b\}$$

$$F^2(a, b) = \{c\}$$

Figure 4: Example of an closure-extension $\mathbf{A}$ where A° is not a substructure.

The monochromatic copy of $\overrightarrow{\mathbf{B}}_1$ will have the property that it is either ordered in the same way as $\overrightarrow{\mathbf{B}}_1$ or the order is reversed. By construction of $\overrightarrow{\mathbf{B}}_0$ it follows that in both alternatives there is a copy of $\overrightarrow{\mathbf{A}}$. $\qquad\square$

In the following we generalize the main idea of this proof (the idea of which goes back to [28]) to classes with non-trivial closures of vertices. In this case we need to define carefully the admissible orderings.

3.1   Admissible orderings

Definition 3.1. Let $\mathbf{A}$ be an L-structure. If $a, b \in A$ we write $a \sim_A b$ if $\mathrm{Cl}_\mathbf{A}(a) = \mathrm{Cl}_\mathbf{A}(b)$. This is an equivalence relation on A and we refer to the classes as the *closure-components* of $\mathbf{A}$. The class containing a will be denoted by $\mathrm{Cc}_\mathbf{A}(a)$.

If $a \in A$ we define the *level* $l_\mathbf{A}(a)$ of a in $\mathbf{A}$ inductively. We say that $l_\mathbf{A}(a) = 0$ in the case where $\mathrm{Cc}_\mathbf{A}(a) = \mathrm{Cl}_\mathbf{A}(a)$; otherwise $l_\mathbf{A}(a) = l_\mathbf{A}(b) + 1$ where b is a vertex of maximal level in $\mathbf{A}$ amongst vertices in $\mathrm{Cl}_\mathbf{A}(a) \setminus \mathrm{Cc}_\mathbf{A}(a)$.

We say that $A \in \mathcal{K}$ is a *closure-extension* at level k if there is a unique closure-component C of vertices of level k in $\mathbf{A}$ and $\mathrm{Cl}_\mathbf{A}(a) = \mathbf{A}$ for every $a \in C$. In this case, we write $A^\circ = A \setminus C$. In other words, every closure of a vertex is a closure-extension.

We say that two closure-components C and C' of $\overrightarrow{\mathbf{A}}$ (or their closures) are *homologous* if $\mathrm{Cl}_\mathbf{A}(C)$ and $\mathrm{Cl}_\mathbf{A}(C')$ are isomorphic and $\mathrm{Cl}_\mathbf{A}(C) \setminus C = \mathrm{Cl}_\mathbf{A}(C') \setminus C'$. Note that the isomorphism must fix each vertex of $\mathrm{Cl}_\mathbf{A}(C) \setminus C$ and, if $C \neq C'$, then $\mathrm{Cl}_\mathbf{A}(C) \setminus C = \mathrm{Cl}_\mathbf{A}(C) \cap \mathrm{Cl}_\mathbf{A}(C')$ is closed in $\mathbf{A}$.

Example. Consider the class $\mathcal{T}$ of forests represented by a single unary function described in Section 1.2. In this class the closure of a vertex is the path to a root vertex. Every vertex thus forms a trivial closure component and its level is determined by the distance to the root vertex. In this particular case a structure is a closure-extension if and only if $\mathbf{A}$ is an oriented path (that is structure on vertex set $\mathbf{A} = \{v_1, v_2, \ldots, v_n\}$ and $F_\mathbf{A}(v_i) = v_{i+1}$ for every $1 \leq i < n$).

Observe also that for a closure-extension $\mathbf{A}$ (depicted in Figure 4), the set of vertices A° is not necessarily closed. Consider a language with a function F^1 from 1-tuples to sets of two vertices and a function F^2 from 2-tuples to singletons. The structure $\mathbf{A}$ with $A = \{a, b, c\}$, $F_{\mathbf{A}}^1(c) = \{a, b\}$, $F_{\mathbf{A}}^2(a, b) = \{c\}$ is an closure-extension of level 1: $l(a) = l(b) = 0$ and $l(c) = 1$, the set A° is $\{a, b\}$, however $\mathrm{Cl}_{\mathbf{A}}\{a, b\} = \{a, b, c\}$.

Suppose $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ are closure-extensions and $\overrightarrow{\mathbf{A}}, \overrightarrow{\mathbf{B}} \in \overrightarrow{\mathcal{K}}$ are orderings. We say that these are *similar* if there is an isomorphism $\alpha : \mathbf{A} \to \mathbf{B}$ which is also order-preserving when seen as a mapping $\alpha : A^\circ \to B^\circ$. This is an equivalence relation and in our admissible orderings, we choose a fixed representative from each similarity-type of ordered closure-extension.

There is some flexibility in the definition and in what follows we shall assume that we have fixed some total ordering $\overrightarrow{\mathbf{A}} \trianglelefteq \overrightarrow{\mathbf{B}}$ between isomorphism types of orderings of closure-extensions such that

S1 $|A| < |B|$ implies $\overrightarrow{\mathbf{A}} \trianglelefteq \overrightarrow{\mathbf{B}}$.

(In particular, $\trianglelefteq$ is a well ordering.)

First we define a preorder of vertices which we will later refine to a linear order. Given two vertices $u \neq v \in \overrightarrow{\mathbf{A}}$ we write $u \preccurlyeq_{\overrightarrow{\mathbf{A}}} v$ if one of the following holds:

P1 $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(u) \trianglelefteq \mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(v)$ and they are not isomorphic;

P2 $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(u)$ is isomorphic to $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(v)$ but $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(u) \setminus \mathrm{Cc}_{\overrightarrow{\mathbf{A}}}(v)$ is lexicographically before $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(v) \setminus \mathrm{Cc}_{\overrightarrow{\mathbf{A}}}(v)$ considering the order $\leq_{\overrightarrow{\mathbf{A}}}$;

P3 $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(u)$ and $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}}(v)$ are homologous closure-extensions.

Note that this is indeed a preorder on the vertices of $\mathbf{A}$. We can now describe our class of admissible orderings.

Definition 3.2. Suppose $\mathcal{K}$ is a free amalgamation class. We say that $\mathcal{O} \subseteq \overrightarrow{\mathcal{K}}$ is a *class of admissible orderings* of structures in $\mathcal{K}$ if the following conditions hold.

A1 If $\mathbf{A} \in \mathcal{K}$, then there is some ordering $\leq_{\overrightarrow{\mathbf{A}}}$ of $\mathbf{A}$ in $\mathcal{O}$.

A2 $\mathcal{O}$ is closed for substructures.

A3 For every $\overrightarrow{\mathbf{A}} \in \mathcal{O}$, the ordering $\leq_{\overrightarrow{\mathbf{A}}}$ refines $\preccurlyeq_{\overrightarrow{\mathbf{A}}}$.

A4 For every $\overrightarrow{\mathbf{A}} \in \mathcal{O}$, the closure-components form linear intervals in $\leq_{\overrightarrow{\mathbf{A}}}$.

A5 For every $\mathbf{B} \in \mathcal{K}$, if $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ is a family of substructures and $\leq$ is a linear order of $A = \cup_{1 \leq i \leq n} A_i$ such that

    (a) $\leq$ satisfies the conclusions of A3 and A4; and

(b) each substructure of **B** contained in A is admissibly ordered by $\leq$;

then there exists $\overrightarrow{\mathbf{B}} \in \mathcal{O}$ such that $\leq_{\overrightarrow{\mathbf{B}}}$ restricted to A is $\leq$.

A6 Suppose that $\overrightarrow{\mathbf{A}}, \overrightarrow{\mathbf{B}} \in \mathcal{O}$ are similar ordered closure-extensions. Then $\overrightarrow{\mathbf{A}}$ is isomorphic to $\overrightarrow{\mathbf{B}}$.

Example. Consider $\mathbf{A} \in \mathcal{T}$ where $\mathcal{T}$ is the class of forests discussed in Section 1.2. Because the closure-extensions are all formed by oriented paths, the order $\trianglelefteq$ requires vertices to be ordered according to their levels (in particular all root vertices come first and can be ordered arbitrarily). The sons of a vertex $v \in \mathbf{A}$ are trivial homologous components and thus they are required to always form an interval and the order amongst these intervals is given by order of their fathers.

There are some choices how to define the admisibility. For example it is possible to order forests in a way that every tree (and recursively every subtree) forms an interval. The particular choice is however not very important as it can be shown that they are all equivalent up to bi-definability (this follows as $\mathcal{O}$ is a Ramsey class, see [20]).

Proposition 3.2. *Suppose $\mathcal{K}$ is a free amalgamation class. Then there is a class $\mathcal{O}$ of admissible orderings of $\mathcal{K}$.*

Proof. We proceed by induction on $\overrightarrow{\mathcal{K}}$ ordered abitrarily in order of increasing number of vertices. In this order, for every $\overrightarrow{\mathbf{C}} \in \overrightarrow{\mathcal{K}}$ we decide if $\overrightarrow{\mathbf{C}} \in \mathcal{O}$ or not by a variant of a greedy algorithm. In the induction step assume that we already decided the presence in $\mathcal{O}$ for every proper substructure of $\overrightarrow{\mathbf{C}}$.

We put $\overrightarrow{\mathbf{C}} \in \mathcal{O}$ if and only if:

O1 $\overrightarrow{\mathbf{C}}$ satisfies A3 and A4,

O2 every proper substructure of $\overrightarrow{\mathbf{C}}$ is in $\mathcal{O}$, and,

O3 if $\overrightarrow{\mathbf{C}}$ is an ordered closure-extension, then there is no similar but non-isomorphic $\overrightarrow{\mathbf{D}} \in \mathcal{O}$.

This finishes the description of $\mathcal{O}$. We verify that the conditions of Definition 3.2 are satisfied.

First we check A5. Assume, to the contrary, that there are $\mathbf{B} \in \mathcal{K}$, substructures $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ and a linear order $\leq$ on A without such an extension. From all counter-examples choose one minimizing $|B|$ and among those, minimize $|B|-|A|$.

We consider three cases:

1. Suppose $\mathbf{B}$ is not a closure-extension and $B = A$. It follows that the order $\leq$ satisfies both O1 and O2 and thus we can put $\leq_{\overrightarrow{\mathbf{B}}}=\leq$ and obtain $\overrightarrow{\mathbf{B}} \in \mathcal{O}$, a contradiction.

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2. Suppose **B** is a closure-extension and $B^\circ = A$. Extend $\leq$ to an order of $\overrightarrow{\mathbf{B}}$ in a way it satisfies O1. In this case $\overrightarrow{\mathbf{B}}$ satisfies O2 because no proper substructure contains $B \setminus B^\circ$. Furthermore, we may assume that O3 holds. This is a contradiction to $\overrightarrow{\mathbf{B}} \notin \mathcal{O}$.

3. Suppose neither of the previous cases apply. Then there is a proper closure-extension $\mathbf{C} \subseteq \mathbf{B}$ such that $C \nsubseteq A$. Denote by $\leq'$ the order $\leq$ restricted to $C \cap A$ and consider the structures $\mathbf{C}_i = \mathbf{A}_i \cap \mathbf{C}$, $1 \leq i \leq n$. From the minimality of the counter-example and because $|C| < |B|$ we know that order of $\leq'$ can be extended to an admissible order $\overrightarrow{\mathbf{C}}$ of $\mathbf{C}$. Now extend to $C \cup A$ the orders $\leq$ and $\leq_{\overrightarrow{\mathbf{C}}}$ in such a way that A3 and A4 are satisfied. This combined order along with the family of substrucutres $\mathbf{C}, \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ contradicts the second assumption about the minimality of the counter-example.

This finishes proof of A5.

Condition A1 is implied by A5. Conditions A2, A3, A4 and A6 follows directly from the construction of $\mathcal{O}$. □

3.2 Proof of Theorem 1.4

Advancing the proof of Theorem 1.4 we show two lemmas which generalize the main ideas of proof of Proposition 3.1.

Lemma 3.3. *Let $\mathcal{R}$ be a Ramsey class of ordered structures. Then for every $\overrightarrow{\mathbf{A}} \in \mathcal{R}$ there exists $\overrightarrow{\mathbf{B}} \in \mathcal{R}$ such that every re-ordering $\overrightarrow{\mathbf{C}} \in \mathcal{R}$ of $\overrightarrow{\mathbf{B}}$ contains a re-ordering $\widetilde{\mathbf{A}}$ of $\overrightarrow{\mathbf{A}}$ such that every two isomorphic substructures of $\overrightarrow{\mathbf{A}}$ are also isomorphic in the order of $\widetilde{\mathbf{A}}$.*

Proof. Let $\overrightarrow{\mathbf{A}}$ be a structure and $\overrightarrow{\mathbf{A}}_0$ a substructure. Denote by n the number of possible re-orderings of $\overrightarrow{\mathbf{A}}_0$ (i.e. $n = |A_0|!$). By the Ramsey property construct $\overrightarrow{\mathbf{B}}_0$ such that $\overrightarrow{\mathbf{B}}_0 \longrightarrow (\overrightarrow{\mathbf{A}})^{\overrightarrow{\mathbf{A}}_0}_n$. Every re-ordering of $\overrightarrow{\mathbf{B}}_0$ induces an n-coloring of copies of $\overrightarrow{\mathbf{A}}_0$ in $\overrightarrow{\mathbf{B}}_0$ and so by the Ramsey property there exists a copy $\widetilde{\mathbf{A}}$ of re-ordered $\overrightarrow{\mathbf{A}}$ in $\overrightarrow{\mathbf{B}}_0$ having the property that all copies of $\overrightarrow{\mathbf{A}}_0$ in $\widetilde{\mathbf{A}}$ are ordered the same way. The statement follows by iterating this argument for every substructure of $\overrightarrow{\mathbf{A}}$. □

Lemma 3.4. *Let $\mathcal{K}$ by a free amalgamation class. Then for every $\overrightarrow{\mathbf{A}} \in \overrightarrow{\mathcal{K}}$ in which every closure-component forms an interval there exists $\overrightarrow{\mathbf{B}}$ such that every re-ordering $\overrightarrow{\mathbf{C}}$ of $\overrightarrow{\mathbf{B}}$ where every closure-component forms an interval contains a copy of a re-ordering of $\overrightarrow{\mathbf{A}}$ where the order between vertices in distinct homologous closure-components is preserved.*

Proof. For simplicity we show the construction for a given $\overrightarrow{\mathbf{A}}$ and two distinct homologous closure-components $C_1 \leq_{\overrightarrow{\mathbf{A}}} C_2$. The full statement can be shown
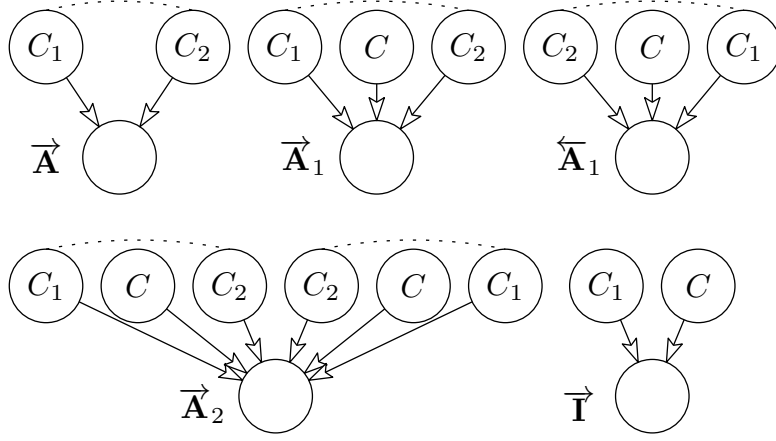
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Figure 5: Construction used in the proof of Lemma 3.4. Pairs of closure components which are not necessarily free are connected by dotted lines.

by iterating the argument for every such pair. The construction is schematically depicted in Figure 5.

First observe that C_1 and C_2 are not in the closure of $\mathrm{Cl}_{\mathbf{A}}(C_1) \setminus C_1 = \mathrm{Cl}_{\mathbf{A}}(C_2) \setminus C_2$ because this would imply $C_1 = C_2$.

Now, by a free amalgamation construct $\overrightarrow{\mathbf{A}}_1$ extending $\overrightarrow{\mathbf{A}}$ with a new closure-component C homologous to both C_1 and C_2 where the linear order is extended so that $C_1 \leq_{\overrightarrow{\mathbf{A}}} C \leq_{\overrightarrow{\mathbf{A}}} C_2$. Denote by $\overleftarrow{\mathbf{A}}_1$ the structure created from $\overrightarrow{\mathbf{A}}_1$ by re-ordering closure-components C_1, C and C_2 so that $C_2 \leq_{\overleftarrow{\mathbf{A}}_1} C \leq_{\overleftarrow{\mathbf{A}}_1} C_1$. Denote by $\overrightarrow{\mathbf{A}}_2$ the free amalgamation of $\overrightarrow{\mathbf{A}}_1$ and $\overleftarrow{\mathbf{A}}_1$ over $\mathrm{Cl}_{\mathbf{A}}(C_1) \setminus C_1$. Because $\mathcal{K}$ is a free amalgamation class we have $\overrightarrow{\mathbf{A}}_1, \overleftarrow{\mathbf{A}}_1, \overrightarrow{\mathbf{A}}_2 \in \overrightarrow{\mathcal{K}}$.

Note that the addition of C into $\mathbf{A}_1$ is necessary only when $\mathrm{Cl}(\overrightarrow{\mathbf{A}}(C_1 \cup C_2))$ is not a result of free amalgamation of two copies of $\mathrm{Cl}_{\overrightarrow{\mathbf{A}}(C_1)}$. In general there may be relations spanning C_1 and C_2 which would make use of the Ramsey argument bellow impossible.

Put $\overrightarrow{\mathbf{I}} = \mathrm{Cl}_{\overrightarrow{\mathbf{A}}_1}(C_1 \cup C)$ and use Theorem 1.3 to find $\overrightarrow{\mathbf{B}} \in \overrightarrow{\mathcal{K}}$ containing $\overrightarrow{\mathbf{A}}_2$ such that $\overrightarrow{\mathbf{B}} \longrightarrow (\overrightarrow{\mathbf{A}}_2)_2^{\overrightarrow{\mathbf{I}}}$. Now every re-ordering $\overrightarrow{\mathbf{C}}$ of $\overrightarrow{\mathbf{B}}$ induces a 2-coloring of the copies of $\overrightarrow{\mathbf{I}}$ in $\overrightarrow{\mathbf{B}}$ which in turn leads to the existence of a copy of a re-ordering of $\overrightarrow{\mathbf{A}}$ where the order of C_1 and C_2 is preserved. $\qquad\square$

Now we are finally ready to prove the main result of this section.

Proof of Theorem 1.4. Given $\overrightarrow{\mathbf{A}} \in \mathcal{O}$ we construct $\overrightarrow{\mathbf{B}} \in \overrightarrow{\mathcal{K}}$ with $\overrightarrow{\mathbf{A}}$ as a substructure such that every admissible re-ordering $\overrightarrow{\mathbf{C}} \in \mathcal{O}$ of $\overrightarrow{\mathbf{B}}$ contains a copy of $\overrightarrow{\mathbf{A}}$.

By application of Lemmas 3.4 and 3.3 it is enough to construct an admissibly ordered $\overrightarrow{\mathbf{B}}_0$ such that that every re-ordering $\overrightarrow{\mathbf{C}}_0 \in \mathcal{O}$ of $\overrightarrow{\mathbf{B}}_0$ with the following two properties contains a copy of $\overrightarrow{\mathbf{A}}$:

(a) the order of vertices in distinct homologous components is preserved; and,

(b) the order in $\vec{\mathbf{C}}_0$ on substructures which are closures of vertices depends only on their isomorphism type in $\vec{\mathbf{B}}_0$.

Given $\vec{\mathbf{A}}$ denote by $\vec{\mathbf{A}}_1$, $\vec{\mathbf{A}}_2,\ldots$, $\vec{\mathbf{A}}_n$ all admissible re-orderings of $\vec{\mathbf{A}}$ having properties (a) and (b). So these structures all have the same domain, and only differ in their orderings. Put $\vec{\mathbf{B}}_0$ to be the disjoint union of $\vec{\mathbf{A}}_1$, $\vec{\mathbf{A}}_2,\ldots$, $\vec{\mathbf{A}}_n$ with the order completed arbitrarily. Let $\vec{\mathbf{C}}_0$ be an admissible reordering of $\vec{\mathbf{B}}_0$ satisfying (a) and (b). Denote by $\alpha_i : \vec{\mathbf{A}} \to \vec{\mathbf{C}}_0$ the map which sends $a \in A$ to $a \in A_i$, for $i \le n$. It is enough to prove the following.

Claim: For all $k \ge 0$, there is some $i \le n$ such that the map α_i preserves the ordering on vertices in $\vec{\mathbf{A}}$ of level at most k.

We prove this by induction on k. Denote by $A|_k$ the set of vertices in A at level $\le k$ (this is not necessarily a substructure or even a structure in $\mathcal{K}$, but it still makes sense to discuss admissible orderings of it as it contains the closures of all of its vertices, and the criteria for being an admissible ordering depend only on these). By setting $A|_{-1} = \emptyset$, we can incorporate the proof for the base case $k = 0$ into the general argument.

Step 1: Every admissible ordering of $\mathbf{A}|_{k-1}$ satisfying (a) and (b) extends to one of $\mathbf{A}|_k$. If $\mathbf{X}$ is the closure of a vertex of level k in $\mathbf{A}$, then any two such closure-extensions differ on $\mathbf{X}$ by a permutation in $\mathrm{Aut}(\mathbf{X}/X^\circ)$ (that is, automorphisms of $\mathbf{X}$ fixing all vertices of level less than k).

Indeed, we need only say how to define the ordering on $\mathbf{X}$. But, given the ordering on X°, this is determined by condition A6 in Definition 3.2, up to the action of $\mathrm{Aut}(\mathbf{X}/X^\circ)$. It is easy to see that the resulting ordering on $A|_k$ is admissible.

Step 2: Suppose the claim holds up to level $k-1$. Let I denote the set of $i \le n$ for which α_i restricted to $A|_{k-1}$ is order-preserving. So this is non-empty. Let $i \in I$ and let $\mathbf{X}$ be as in Step 1.

There is $\beta_i \in \mathrm{Aut}(\mathbf{X}/X^\circ)$ such that $\alpha_i \circ \beta_i$ preserves the ordering on $\mathbf{X}$ and so is an isomorphism between $\vec{\mathbf{X}}$, the structure on $\mathbf{X}$ in $\vec{\mathbf{A}}$ and the substructure $\alpha_i(\mathbf{X})$ in $\vec{\mathbf{C}}_0$. By Step 1, there is some $j \le n$ such that the map β_i^{-1}, regarded as a map from $\vec{\mathbf{A_i}}$ to $\vec{\mathbf{A_j}}$ (in $\vec{\mathbf{B}}_0$) is order-preserving and all vertices in $A|_{k-1}$ have the same ordering in $\vec{\mathbf{A}}_i$ and $\vec{\mathbf{A}}_j$. By condition (b), it follows that this map between the corresponding subsets of $\vec{\mathbf{C}}_0$ is order-preserving (as all orderings are determined by what happens in closures of vertices). Thus $j \in I$ and α_j is order-preserving on X. Repeating this argument for other vertices at level k, we complete the inductive step.

We have verified that $\mathcal{O}$ has the ordering property. The Ramsey property follows from Theorem 1.3 and A5: Given $\vec{\mathbf{A}}, \vec{\mathbf{B}} \in \mathcal{O}$ and $\vec{\mathbf{C}} \in \vec{\mathcal{K}}$ such that $\vec{\mathbf{C}} \longrightarrow (\vec{\mathbf{B}})_2^{\vec{\mathbf{A}}}$. Construct a new order $\le$ of $\vec{\mathbf{C}}$ in a way that $\le$ agrees with $\le_{\mathbf{C}}$ on every copy of $\vec{\mathbf{B}}$ in $\vec{\mathbf{C}}$. Complete $\le$ so it satsifies A3 and A4. By A5 it follows that $\mathbf{C}$ ordered by $\le$ is in $\mathcal{O}$. $\qquad\square$

4 Irreducible substructure faithful EPPA

The proof of Theorem 1.7 is a variant of the proof of clique-faithful EPPA by Hodkinson and Otto [16] combined with a proof of EPPA for purely functional language from [10]. As in [16] we build on the following construction giving EPPA for relational structures and we verify coherency as in [31, 32].

Theorem 4.1 (Herwig [12], coherency verified by Solecki [34])**.** *For any relational language L, the class of all finite L-structures has coherent EPPA.*

To apply this construction to structures with functions we will temporarily interpret functions as relational symbols.

Definition 4.1. Suppose L is a language. Given an L-structure $\mathbf{A}$ we denote by $\mathbf{A}^-$ its *relational reduct* constructed as follows. The language L^- of $\mathbf{A}^-$ is a relational language containing all relational symbols of $\mathbf{A}$ and additionally containing for every function symbol $F \in L_{\mathcal{F}}$ a relation symbol $R^F \in L_{\mathcal{F}}^-$ of arity $a(R^F) = d(F)+1$. The vertex set of $\mathbf{A}$ is same as the vertex set of $\mathbf{A}^-$. For every $R \in L_{\mathcal{R}}$ we have $R_{\mathbf{A}} = R_{\mathbf{A}^-}$ and for every $F \in L_{\mathcal{F}}$ it holds that $\vec{t} \in R_{\mathbf{A}^-}^F$ if and only if $F_{\mathbf{A}}(\vec{t}_1) = S$ where $\vec{t}_1$ is the tuple of the first $d(F)$ elements of $\vec{t}$ and the last vertex of $\vec{t}$ is in S (recall: our functions assign subsets to tuples).

Note that in the above, the structures $\mathbf{A}$ and $\mathbf{A}^-$ have the same automorphisms and any partial automorphism of $\mathbf{A}$ is an automorphism of $\mathbf{A}^-$.

The difference between our proof and that in [16] and [32] (which give clique faitful EPPA for structures in relational lanugages) is that whereas in these papers, the vertices in the EPPA-extension are of the form (v, χ_v) (as in the notation below), our vertices will actually be sets of such vertices (carrying an L-structure).

Proof of Theorem 1.7. Given $\mathbf{A}$ we invoke Theorem 4.1 to obtain a coherent EPPA-extension $\mathbf{B}^-$ of $\mathbf{A}^-$ (with respect to all partial isomorphisms of $\mathbf{A}^-$). We use the following terminology, following an exposition by Hodkinson of the argument in [15].

A set $S \subseteq B^-$ is called *small* if there is some $g \in \mathrm{Aut}(\mathbf{B}^-)$ such that $g(S) \subseteq A$. Otherwise S is called *big*. Denote by $\mathcal{U}$ the set of all big subsets of B^- and note that this is preserved by $\mathrm{Aut}(\mathbf{B}^-)$. Given $b \in B^-$ a map $\chi : \mathcal{U} \to \mathbb{N}$ is a *b-valuation function* if $\chi(S) = 0$ for all $b \notin S \in \mathcal{U}$ and $1 \le \chi(S) < |S|$ otherwise.

Given vertices $u, v \in B^-$ and their valuation functions, χ_u and χ_v we say that (u, χ_u) and (v, χ_v) are *generic* if either $(u, \chi_u) = (v, \chi_v)$, or $u \ne v$ and for every $S \in \mathcal{U}$ such that $u, v \in S$ it holds that $\chi_u(S) \ne \chi_v(S)$.

The key construction in the proof is the following "local covering" construction of a *b-valuation L-structure* $\mathbf{V}_b$. Fix $b \in B^-$ and an automorphism $\alpha : B^- \to B^-$ such that $\alpha(b) \in \mathbf{A}^-$ (we can assume such an automorphism

always exists — all other vertices can be removed from $\mathbf{B}^-$). Now consider the L-substructure $\mathbf{V}_b^\alpha = \mathrm{Cl}_{\mathbf{A}}(\alpha(b))$ of $\mathbf{A}$. Suppose that for every $v \in \alpha^{-1}(V_b^\alpha)$ we have a v-valuation function χ_v such that the assigned valuations are generic for every pair of vertices in $\alpha^{-1}(V_b^\alpha)$. (Such a choice of valuation functions always exists and we will show how to obtain it later.) Denote by V_b the set of all pairs (v, χ_v), $v \in \alpha^{-1}(V_b^\alpha)$. On the set V_b we consider the L-structure $\mathbf{V}_b$, called a *b-valuation*, which is defined in such a way that the composition of mappings α and $\pi(v, \chi_v) = v$ forms an embedding $\alpha \circ \pi : \mathbf{V}_b \to \mathbf{V}_b^\alpha$. (This is a standard construction, we use the 1–1 mapping $\alpha \circ \pi$ to pull back the structure $\mathbf{V}_b^\alpha$ to V_b; note that the structure here does not depend on the choice of α.) Observe that then $\mathrm{Cl}_{\mathbf{V}_b}((b, \chi_b)) = \mathbf{V}_b$.

Notice that for every b there are multiple choices of b-valuations $\mathbf{V}_b$ (depending on particular choice of valuation functions assigned to vertices, but not depending on the choice of α). The sets V_b and structures $\mathbf{V}_b$ will form a "cover of $\mathbf{B}^-$" and we find it convenient to make the following definitions.

Definition 4.2. Recalling that all functions of L are unary, we say that a pair of valuations $\mathbf{V}_a$ and $\mathbf{V}_b$ is *generic* if

(i) every pair of vertices $(u, \chi_u) \in V_a$ and $(v, \chi_v) \in V_b$ is generic;

(ii) for every $(u, \chi_u) \in V_a$ and $(v, \chi_v) \in V_b$ and $F \in L_{\mathcal{F}}$ it holds that

    (a) if $(u, v) \in R_{\mathbf{B}^-}^F$, then $(v, \chi_v) \in V_a$ and

    (b) if $(v, u) \in R_{\mathbf{B}^-}^F$, then $(u, \chi_u) \in V_b$;

(iii) if $(u, \chi_u) \in V_a \cap V_b$, then $\mathrm{Cl}_{\mathbf{V}_a}((u, \chi_u)) = \mathrm{Cl}_{\mathbf{V}_b}((u, \chi_u))$.

We also say that a set S of valuations is *generic* if every pair of valuations in S is generic.

Now we construct an L-structure $\mathbf{C}$:

1. The vertices of $\mathbf{C}$ are all b-valuation L-structures $\mathbf{V}_b$, for $b \in B^-$.

2. For every relation $R \in L_{\mathcal{F}}$ put $(\mathbf{V}_{v_1}, \mathbf{V}_{v_2}, \dots, \mathbf{V}_{v_{a(R)}}) \in R_{\mathbf{C}}$ if and only if $(v_1, v_2, \dots, v_{a(R)}) \in R_{\mathbf{B}^-}$ and the set $\{\mathbf{V}_{v_i} : 1 \le i \le a(R)\}$ is generic.

3. For every function $F \in L_{\mathcal{F}}$ put $F_{\mathbf{C}}(\mathbf{V}_{v_1}) = \{\mathbf{V}_{v_2}, \mathbf{V}_{v_3}, \dots, \mathbf{V}_{v_{r(F)+1}}\}$ if and only if $(v_1, v_l) \in R_{\mathbf{B}^-}^F$ for every $2 \le l \le r(F) + 1$ and the set $\{\mathbf{V}_{v_i} : 1 \le i \le r(F) + 1\}$ is generic.

First we verify that $\mathbf{C}$ is indeed an L-structure. It will suffice to show that if $F \in L_{\mathcal{F}}$, $\mathbf{V}_v \in \mathbf{C}$ and $F(v) = \{u_1, \dots, u_s\}$ (where $s = r(F)$), then there are unique $\mathbf{V}_{u_1}, \dots, \mathbf{V}_{u_s} \in \mathbf{C}$ with $\{\mathbf{V}_v, \mathbf{V}_{u_1}, \dots, \mathbf{V}_{u_s}\}$ generic (and therefore $F_{\mathbf{C}}(\mathbf{V}_v) = \{\mathbf{V}_{u_1}, \dots, \mathbf{V}_{u_s}\}$). But as $(v, u_i) \in R_{\mathbf{B}^-}^F$, genericity of $\mathbf{V}_v, \mathbf{V}_{u_i}$ implies that $\mathbf{V}_{u_i} \subseteq \mathbf{V}_v$. So $\mathbf{V}_{u_i} = \mathrm{Cl}_{\mathbf{V}_v}(u_i, \chi_{u_i})$, where $(u_i, \chi_{u_i}) \in \mathbf{V}_v$.

Next we give an embedding $\phi : \mathbf{A} \to \mathbf{C}$ with generic image. For every big set $S \in \mathcal{U}$ choose $f_S : S \to \{1, 2, \ldots, |S| - 1\}$ to be a function such that $f_S(v) > 0$ if and only if $v \in A \cap S$ and for every pair of vertices $u, v \in A \cap S$ it holds that $f_S(u) \neq f_S(v)$. Such a function exists because $A \cap S$ is always a proper subset of S. Given a vertex $a \in A$ we put $\phi(a)$ to be an a-valuation constructed from $\mathrm{Cl}_{\mathbf{A}}(a)$ by mapping every vertex $v \in \mathrm{Cl}_{\mathbf{A}}(a)$ to (v, χ_v) where $\chi_v(S) = f_S(v)$. It is easy to verify that this is indeed an embedding from $\mathbf{A}$ to $\mathbf{C}$ and $\phi(\mathbf{A})$ is generic.

We aim to show that $\mathbf{C}$ is an EPPA-extension of $\phi(\mathbf{A})$. We first take time to prove a lemma which will allow us to use the fact that $\mathbf{B}^-$ is an EPPA-extension of $\mathbf{A}^-$. Denote by $\mathcal{V}$ the union of all vertex sets of $\mathbf{V}_v \in \mathbf{C}$. If $g \in \mathrm{Aut}(\mathbf{B}^-)$, we say that the partial map $q : \mathcal{V} \to \mathcal{V}$ is g-compatible if for all $(a, \chi) \in \mathrm{Dom}(q)$ there exists a $g(a)$-valuation function χ' such that $q((a, \chi)) = (g(a), \chi')$. Let $g \in \mathrm{Aut}(\mathbf{B}^-)$ and $p : \mathbf{C} \to \mathbf{C}$ be a partial automorphism. We say that p is g-compatible if there exists a g-compatible map $q : \mathcal{V} \to \mathcal{V}$ such that for all $\mathbf{V}_v \in \mathrm{Dom}(p)$ q is an isomorphism of $\mathbf{V}_v$ and $p(\mathbf{V}_v)$. Denote by π the homomorphism (projection) $\mathbf{C}^- \to \mathbf{B}^-$ defined by $\pi(\mathbf{V}_v) = v$.

Lemma 4.2. *Let $p : \mathbf{C} \to \mathbf{C}$ be a partial automorphism with generic domain and range, $g \in \mathrm{Aut}(\mathbf{B}^-)$, and suppose that p is g-compatible. Then p extends to some g-compatible $\hat{p} \in \mathrm{Aut}(\mathbf{C})$.*

Proof. As $\mathrm{Dom}(p)$ is generic, for every $v \in \pi(\mathrm{Dom}(p))$ there is precisely one v-valuation function χ_v such that the pair (v, χ_v) is a vertex of some valuation $V_b \in \mathrm{Dom}(p)$. Denote by D the set of all such pairs (so $D = \bigcup \mathrm{Dom}(p)$). The same is true for the range and denote by R all pairs appearing as vertices in valuations in $p(\mathrm{Dom}(p))$. It follows that p uniquely defines a g-compatible map $q : D \to R$. Fix a big set $S \in \mathcal{U}$. Then the set of pairs

$$\left\{ \left(\chi_b(S), \chi'_{g(b)}(g(S)) \right) : (b, \chi_b) \in D, \, q(b, \chi_b) = (g(b), \chi'_{g(b)}) \right\}$$

is the graph of a partial permutation of $\{0, 1, 2, 3, \ldots, |S| - 1\}$ fixing 0 if defined on it. Extend it to a permutation θ_S^p of $\{0, 1, 2, 3, \ldots, |S| - 1\}$ fixing 0.

Now we define $\hat{q} : \mathcal{V} \to \mathcal{V}$ by mapping $(b, \chi) \in \mathcal{V}$ to $(g(b), \chi')$ such that $\chi'(g(S))) = \theta_S^p(\chi(S))$ and $\hat{p} : \mathbf{V}_v \to \mathbf{V}_{g(v)}$ where $\mathbf{V}_{g(v)}$ is created from $\mathbf{V}_v$ by mapping every vertex $(b, \chi) \in V_v$ to $\hat{q}(b, \chi)$.

It is easy to verify that $\hat{q}$ is a well defined permutation of $\mathcal{V}$ which extends q and is g-compatible. Moreover it preserves the relation of genericity between elements of $\mathcal{V}$. Therefore also $\hat{p}$ is a well-defined permutation of C which preserves generic sets, extends p and is g-compatible. Consequently $\hat{p}$ is an automorphism of $\mathbf{C}$. $\square$

By Lemma 4.2 the extension property of $\mathbf{C}$ for partial isomorphisms of $\phi(\mathbf{A})$ follows easily. Let p be a partial isomorphism of $\phi(\mathbf{A})$. We extend it to $\hat{p} \in \mathrm{Aut}(\mathbf{C})$. First extend $\phi^{-1} \circ p$ (which is an partial automorphism of $\mathbf{A}$) to

automorphism $g \in \text{Aut}(\mathbf{B}^-)$. Clearly p is g-compatible and because domain and range are generic, by Lemma 4.2 p extends to g-compatible $\hat{p} \in \text{Aut}(\mathbf{C})$. This shows that $\mathbf{C}$ is indeed an extension of $\phi(\mathbf{A})$.

For coherence, we use the same argument as in [31, 32]. Given a coherent triple (f_0, g_0, h_0) of partial automorphsims of $\mathbf{A}$ we first extend this to a coherent triple (f, g, h) of automorphisms of $\mathbf{B}^-$. We let (f_1, g_1, h_1) be the coherent triple of partial automorphisms of $\phi(\mathbf{A})$ induced by (f_0, g_0, h_0). Using Lemma 4.2 we extend f_1 to an f-compatible $\hat{f} \in \text{Aut}(\mathbf{C})$. Similarly we obtain extensions $\hat{g}$, $\hat{h}$ of g, h. In order to ensure that the triple $(\hat{f}, \hat{g}, \hat{h})$ is coherent, we only need to ensure that, in the proof of Lemma 4.2, the permutations θ_S^p can be chosen coherently. More precisely, we want to ensure that $\theta_S^g \theta_S^f = \theta_S^h$. As in [32], if we extend any partial permutation α on $\{1, \ldots, s\}$ to a permutation by mapping $\{1, \ldots, s\} \setminus \text{Dom}(\alpha)$ to $\{1, \ldots, s\} \setminus \alpha(\text{Dom}(\alpha))$ in an order-preserving way, then we obtain the required coherence.

Finally we verify that $\mathbf{C}$ is faithful for irreducible substructures. Let $\mathbf{D}$ be an irreducible substructure of $\mathbf{C}$. We first show that $\mathbf{D}$ is generic. Suppose not and that $\mathbf{V}_a, \mathbf{V}_b \in D$ form a non-generic pair of vertices. Let $\mathbf{E}_a = \{\mathbf{V}_v \in D : \mathbf{V}_a \notin \text{Cl}_{\mathbf{D}}(\mathbf{V}_v)\}$. As closures are unary, this is a (proper) substructure of $\mathbf{D}$. Similarly define $\mathbf{E}_b$. Note that $\mathbf{E}_a \cup \mathbf{E}_b = \mathbf{D}$: otherwise, there is $\mathbf{V}_v \in \mathbf{D}$ with $\mathbf{V}_a, \mathbf{V}_b \in \text{Cl}_{\mathbf{D}}(\mathbf{V}_v)$ and then $\mathbf{V}_a, \mathbf{V}_b \subseteq \mathbf{V}_v$, so form a generic pair. Moreover, no relation of $\mathbf{C}$ can involve a vertex $\mathbf{V}_u \in \mathbf{E}_a \setminus \mathbf{E}_b$ and a vertex $\mathbf{V}_v \in \mathbf{E}_b \setminus \mathbf{E}_a$ as $\mathbf{V}_b \subseteq \mathbf{V}_u$ and $\mathbf{V}_a \subseteq \mathbf{V}_v$, which implies that $\mathbf{V}_u, \mathbf{V}_v$ is not a generic pair. Thus $\mathbf{D}$ is a free amalgam of the substructures $\mathbf{E}_a$ and $\mathbf{E}_b$, which is a contradiction to its irreducibility. So $\mathbf{D}$ is generic.

Because $\mathbf{D}$ is generic it follows that $S = \pi(D)$ is small. Indeed, for each $u \in S$, there is a u-valuation χ_u such that the set of pairs $\{(u, \chi_u) : u \in S\}$ is generic. If S were big, this would imply that $\{\chi_u(S) : u \in S\}$ has size $|S|$, which is impossible (its elements j satisfy $1 \leq j < |S|$).

It follows that there is $g \in \text{Aut}(\mathbf{B}^-)$ such that $g(\pi(\mathbf{D})) \subseteq A$. The map $p : \mathbf{D} \to \phi(\mathbf{A})$ given by $p(\mathbf{V}_u) = \phi(g(u))$ is a g-compatible partial automorphism of $\mathbf{C}$ with generic domain and range. By Lemma 4.2, p extends to $\hat{p} \in \text{Aut}(\mathbf{C})$ and $\hat{p}(\mathbf{D}) \subseteq \phi(\mathbf{A})$. This completes the proof that $\mathbf{C}$ is faithful for irreducible substructures. □

Remark. The construction above adds an extra tool to the existing constructions of EPPA-extensions and can be thus used as an additional layer in the construction of EPPA-extensions for non-free amalgamation classes based on application of Herwig-Lascar theorem [13, 34, 30]. An example of such application is given in [2] giving EPPA for some classes of antipodal metric spaces.

5   Applications

In this section we discuss how some previously-studied classes of structures can be viewed naturally as free amalgamation classes of structures with partial functions.

Before doing this, we mention an alternative viewpoint for classes of structures with closures used in [10]. In the following examples we will show how this is related to our definitions.

Consider a class $\mathcal{K}$ of finite L-structures, closed under isomorphisms, and a distinguished class $\sqsubseteq$ of embeddings between elements of $\mathcal{K}$, called *strong embeddings*. We shall assume $\sqsubseteq$ is closed under composition and contains all isomorphisms. In this case, we refer to $(\mathcal{K}; \sqsubseteq)$ as a *strong class*. If $\mathbf{A}$ is a substructure of $\mathbf{B} \in \mathcal{K}$ and the inclusion map $\mathbf{A} \to \mathbf{B}$ is in $\sqsubseteq$, then we say that $\mathbf{A}$ is a *strong substructure* of $\mathbf{B}$ and write $\mathbf{A} \sqsubseteq \mathbf{B}$. In the other words, a strong class is a subcategory of $\mathcal{K}$ with the strong embeddings.

The Ramsey property and amalgamation property can be defined analogously to the Ramsey property and amalgamation property of classes of L-structures, but considering only strong substructures and strong embedding. Most of the Fraïssé theory remains unaffected in this setting (see [10] for more details).

5.1 k-orientations

For a fixed natural number k, a *k-orientation* is an oriented (that is, directed) graph such that the out-degree of every vertex is at most k. We say that a substructure $G_1 = (V_1, E_1)$ of a k-orientation $G_2 = (V_2, E_2)$ is *successor closed* if there is no edge from V_1 to $V_2 \setminus V_1$ in G_2.

Denote by $\mathcal{D}_k$ the class of all finite k-orientations. This is a hereditary class closed for free amalgamation over successor-closed subgraphs and thus the successor-closedness plays the rôle of strong substructure, so $\mathcal{D}_k$ can be considered as a class with corresponding strong embeddings. We show how to turn $\mathcal{D}_k$ into a free amalgamation class in the sense of Definition 1.1.

Given an oriented graph $G = (V, E) \in \mathcal{D}_k$ denote by $\mathbf{G}^+$ the structure with vertex set V and (partial) unary functions $F^1, F^2, \ldots, F^k$. The function F^i, $1 \le i \le k$, is defined for every vertex of out-degree i and maps the vertex to all vertices in its out-neighborhood. Denote by $\mathcal{D}_k^+$ the class of all structures $\mathbf{G}^+$ for $G \in \mathcal{D}_k$. Because $\mathbf{G}_1^+$ is a substructure of $\mathbf{G}_2^+$ if and only if G_1 is successor closed in G_2 it follows that $\mathcal{D}_k^+$ is a free amalgamation class. We immediately obtain:

Theorem 5.1. *The class $\mathcal{D}_k^+$ has the extension property for partial automorphisms. The class $\overrightarrow{\mathcal{D}}_k^+$ is Ramsey and there exists a class $\mathcal{O}_k \subseteq \overrightarrow{\mathcal{D}}_k^+$ with the ordering property (with respect to $\mathcal{D}_k^+$).*
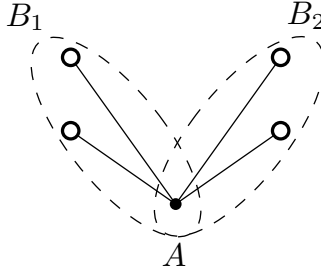
Figure 6: An amalgamation of $B_1, B_2 \in \mathcal{C}_2$ over A.

Let us briefly discuss what is the structure of $\mathcal{O}_k$. Given $\overrightarrow{\mathbf{A}} \in \mathcal{O}_k$, the closure-components (recall Definition 3.1) of $\overrightarrow{\mathbf{A}}$ corresponds to strongly connected components in the underlying oriented graph and $\overrightarrow{\mathbf{A}}$ is an ordered closure-extension of level 0 if and only if the underlying graph is strongly connected. More generally $\overrightarrow{\mathbf{A}}$ is an ordered closure-extension of level k if it contains a single strongly connected component C of level k and all other vertices of $\overrightarrow{\mathbf{A}}$ are reachable from C via an oriented path. Condition A6 of Definition 3.2 thus requires that the ordering of C is determined by the isomorphism type of $\mathbf{A}$ (the underlying oriented graph) and the ordering of $A \setminus C$. Thus in $\mathcal{O}_k$, vertices are ordered primarily by the number of vertices in their closure. Every closure-component forms an interval where the order within this interval is fixed by the similarity type of corresponding closure-extension. The relative order of closure-components is given by their isomorphism type and the ordering of closure-components reachable from them. This can be seen as a generalization of the order of oriented forests described in Section 3.

This can be seen as the most elementary use of Theorems 1.3, 1.4, and 1.7, but it has important consequences. Denote by $\mathcal{C}_k$ the undirected reducts of oriented graphs in $\mathcal{D}_k$, that is, the class of all unoriented graphs which can be oriented to an k-orientation. Given a graph $G = (V, E)$, its *predimension* is $\delta(G) = k|V| - |E|$. It is the heart of Hrushovski predimension construction that the class $\mathcal{C}_k$ forms a free amalgamation class for the following notion of strong subgraph. Given a graph $G \in \mathcal{C}_k$ its subgraph H is a *self sufficient* or *strong subgraph* if for every subgraph H' of G containing H it holds that $\delta(H) \leq \delta(H')$.

The connection between the Hrushovski predimension construction and orientability follows by the Marriage theorem and was first introduced in [8, 9]. Its consequences in Ramsey theory are the main topic of [10] and they are out of scope of this paper. We however point out why this free amalgamation class over strong subgraphs does not translate to a free amalgamation class when enriched by partial functions representing the smallest self-sufficient subgraph of a given set. Consider a graph in $\mathcal{C}_2$ created as amalgamation depicted in Figure 6. While in both B_1 and B_2 the vertices denoted by circles forms a self-sufficient substructures, it is not the case in the free amalgamation. The predimension of the 4 independent vertices is 8, while the predimension of the whole amalgam is 6. It follows that in order to represent self-sufficient substructures by means
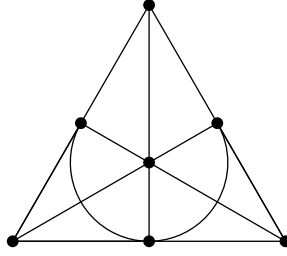
Figure 7: Fano plane (Steiner (2,3)-system).

of partial functions, a new function from the vertices denoted by circles would need to be added. This makes the amalgamation non-free in our representation and this is the reason why additional information about orientation of the edges is needed.

5.2 Steiner systems

It was was established in [3] that the class of finite partial Steiner systems is Ramsey with respect to strong subsystems. Moreover, the ordering property follows from techniques of [28]. We derive both results by a re-interpretation of partial Steiner systems as a free amalgamation class in a functional language.

This is an example where non-unary functions are necessary.

For fixed integers $r \geq t \geq 2$, by a *partial Steiner (r,t)-system* we mean an r-uniform hypergraph $G = (V, E)$ with the property that ever t-element subset of V is contained in at most one edge of G (if there is exactly one such edge, we have a Steiner system). Abusing terminology somewhat, we shall refer to this simply as a Steiner (r,t)-system. Given two Steiner (r,t)-systems G and H, we say that G is a *strongly induced subsytem* of H if

1. G is an induced subhypergraph of H; and,

2. every hyperedge of H which is not a hyperedge of G intersects G in at most $t-1$ vertices.

In [3] the Ramsey property was formulated with respect to strongly induced subsystems. We, equivalently, use partial functions to represent this.

Definition 5.1. Denote by $\mathcal{S}_{r,t}$ the class of all finite structures **A** with one partial function F from t-tuples to r-sets with the following properties:

1. Every t-tuple $\vec{x} \in \text{Dom}(F_{\mathbf{A}})$ has no repeated vertices.

2. For every $\vec{x} \in \text{Dom}(F_{\mathbf{A}})$ it holds that every vertex of $\vec{x}$ is in $F_{\mathbf{A}}(\vec{x})$ and every t-tuple $\vec{x}_2$ of distinct vertices of $F_{\mathbf{A}}(\vec{x})$ is in $\text{Dom}(F_{\mathbf{A}})$ and $F_{\mathbf{A}}(\vec{x}) = F_{\mathbf{A}}(\vec{x}_2)$.

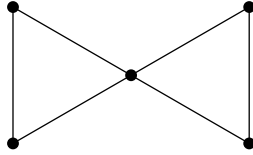It is easy to see that $\mathcal{S}_{r,t}$ is a free amalgamation class.

Figure 8: The bowtie graph.

Given a Steiner (r,t)-system $G = (V, E)$ we can interpret it as a structure $\mathbf{S}_G \in \mathcal{S}_{r,t}$ with vertex set V and function $F(\vec{x})$ defined for every t-tuple $\vec{x}$ of distinct vertices such that there is hyperedge $A \in E$ containing all vertices of $\vec{x}$. In this case we put $F(\vec{x}) = A$.

Observe that if G is a strong subsystem of H if and only if $\mathbf{S}_G$ is a substructure of $\mathbf{S}_H$. It follows that Steiner (r,t)-systems are in 1–1 correspondence to structures $\mathbf{S}_G$ and moreover this correspondence maps subsystems to substructures.

We obtain an alternative proof of the following main result of [3]:

Theorem 5.2. *The class $\overrightarrow{\mathcal{S}}_{r,t}$ is a Ramsey class with the ordering property.*

Proof. The Ramsey property follows directly from Theorem 1.3. For the ordering property, note that single vertices are closed in structures in $\mathcal{S}_{r,t}$, so all orderings are admissible, in the sense of Theorem 1.4. $\square$

We remark that the extension property for partial automorphisms is, to our knowledge, presently open for the class of partial Steiner systems. Of course, our result does not apply in this case, as the function introduced is not unary.

5.3 Bowtie-free graphs

A *bowtie B* is a graph consisting of two triangles with one vertex identified (see Figure 8). A graph G is *bowtie-free* if there is no monomorphism from B to G. The existence of a universal graph in the class of all countable bowtie-free graphs was shown in [22]. The paper [6] gave a far reaching generalization by giving a condition for the existence of ω-categorical universal graphs for classes defined by forbidden monomorphisms (which we refer to as to as *Cherlin-Shelah-Shi classes*). This led to several new classes being identified [7], [5], [4]. Bowtie-free graphs represent a key example of a class that is not a free amalgamation class by itself, but can be turned into one by means of unary functions. This analysis was carried in [18] where we gave an explicit characterisation of the ultrahomogeneous lift and the Ramseyness of this lift. The presentation can be greatly simplified by considering structures with partial functions and moreover we show the extension property for partial automorphisms (See also [31] for related results on ample generics).

While not all Cherlin-Shelah-Shi classes give rise to free amalgamation classes (see the more detailed analysis in [19]), what follows can be generalized to many of the other block-path examples given by [5] and [4].

We review the main observations about the structure of bowtie-free graphs from [18]. For completeness we include the (easy) proofs.

Definition 5.2 (Chimneys)**.** For $n \geq 2$, an *n-chimney graph*, Ch_n, is a free amalgamation of n triangles over one common edge. A *chimney graph* is any graph Ch_n for some $n \geq 2$.

Chimneys together with K_4 (a clique on 4 vertices) will form the only components of bowtie-free graphs formed by triangles. The assumption $n \geq 2$ for chimney is a technical assumption to avoid isolated triangles. Note also that Ch_2 is not an induced subgraph of K_4.

Definition 5.3 (Good bowtie-free graphs)**.** A bowtie-free graph $G = (V, E)$ is *good* if every vertex is contained either in a copy of chimney or a copy of the complete graph K_4.

The structure of bowtie-free graphs is captured by means of the following three lemmas:

Lemma 5.3 ([18])**.** *Every bowtie-free graph G is an induced subgraph of some good bowtie-free graph G'.*

Proof. The graph G can be extended in the following way:

1. For every vertex v not contained in a triangle add a new copy of Ch_2 and identify the vertex v with one of the vertices of Ch_2.

2. For every triangle v_1, v_2, v_3 that is not part of a 2-chimney nor a K_4, add a new vertex v_4 and the triangle v_1, v_2, v_4 turning the original triangle into Ch_2.

It is easy to see that step 1. cannot introduce a new bowtie.

Assume, to the contrary, that step 2. introduced a new bowtie. Further assume that v_1 is the unique vertex of degree 4 of this new bowtie and consequently there is another triangle on vertex v_1 in G. Because G is bowtie-free, this triangle must share a common edge with the triangle v_1, v_2, v_3 and therefore the triangle v_1, v_2, v_3 is already part of a K_4 or a 2-chimney in the original graph G. A contradiction. $\square$

Now we are ready to describe how to turn the class of bowtie-free graphs into a free amalgamation class. Our language L will consist of one binary relation R and unary functions F^1, F^2 and F^3 with arities $r(F^1) = 1$, $r(F^2) = 2$, $r(F^3) = 3$.

For every good bowtie-free graph $G = (V, E)$ denote by $\mathbf{G}^+$ the L-structure with vertex set V and relations and functions defined as follows:

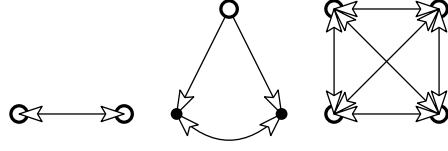1. $(u, v) \in R_{\mathbf{G}}$ if and only if $\{u, v\}$ is edge of G.

Figure 9: All isomorphism types of closure-extensions in $\mathcal{B}$.

2. $F^1_{\mathbf{G}}(v) = u$ if and only if $\{u, v\}$ is contained in at least two triangles of a chimney.

3. $F^2_{\mathbf{G}}(v) = \{u_1, u_2\}$ if and only if $\{v, u_1, u_2\}$ is a triangle of a chimney and v is not contained in multiple triangles.

4. $F^3_{\mathbf{G}}(v) = \{u_1, u_2, u_3\}$ if and only if $\{v, u_1, u_2, u_3\}$ forms a 4-clique in G.

Denote by $\mathcal{B}$ the class of all substructures of structures $\mathbf{G}^+$ where G is a good bowtie-free graph.

Theorem 5.4. *$\mathcal{B}$ is a free amalgamation class.*

Proof. Let $\mathbf{A}, \mathbf{B}, \mathbf{B}' \in \mathcal{B}$. Assume that $\mathbf{A}$ is a substructure of both $\mathbf{B}$ and $\mathbf{B}'$. We show that the free amalgamation $\mathbf{C}$ of $\mathbf{B_1}$ and $\mathbf{B_2}$ over $\mathbf{A}$ is in $\mathcal{B}$.

There are good bowtie free graphs G_1 and G_2 such that $\mathbf{B} \subseteq \mathbf{G}^+_1$ and $\mathbf{B}' \subseteq \mathbf{G}^+_2$. We claim that that the free amalgamation H of G_1 and G_2 over A is a good bowtie-free graph and $\mathbf{H}^+$ is the free amlgam of $\mathbf{G}^+_1$ and $\mathbf{G}^+_2$ over $\mathbf{A}$. As $\mathbf{C}$ is a substructure of $\mathbf{H}^+$, it then follows that $\mathbf{C} \in \mathcal{B}$.

Because $\mathbf{A}$ is a substructure of both $\mathbf{G}^+_1$ and $\mathbf{G}^+_2$, the functions F^1 and F^2 ensure that the free amalgamation preserves the structure of chimneys: if a vertex of a chimney in G_1 is identified with a vertex of a chimney in G_2 (because it is in A) then also the bases (i.e. the edges in multiple triangles) of these chimneys are contained in A, so are identified in H and the result is again a chimney. Similarly F^3 makes sure that a 4-clique containing a vertex of A is in A. Finally free amalgamation cannot introduce any new triangles and thus the free amalgamation is a good bowtie-free graph H and $\mathbf{H}^+$ is the free amalgam of $\mathbf{G}^+_1$ and $\mathbf{G}^+_2$ over $\mathbf{A}$. $\qquad\square$

Corollary 5.5. *The class $\mathcal{B}$ has the irreducible-structure faithful extension property for partial automorphisms; $\overrightarrow{\mathcal{B}}$ is a Ramsey class and there is $\mathcal{B}' \subseteq \overrightarrow{\mathcal{B}}$ with the ordering property (with respect to $\mathcal{B}$).*

The class $\mathcal{B}'$ can be easily derived from the Definition 3.2. There are only three types of closure-extensions in $\mathcal{B}$ depicted in Figure 9 (with arrows representing functions F^1, F^2 and F^3 and circles denoting the vertices of maximal level). It follows that vertices are ordered by size of their closures (here we make use of the definition of $\preceq$ refining the order given by number of vertices). That is vertices in bases of chimney are first, vertices in the top of chimneys next and

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vertices in 4-cliques last. Every vertex-closure forms an interval. Pair of vertices of level 1 (in the the top of chimneys) form homologous extensions if and only if they belong to the same chimney. It follows that for every chimney the set of its top vertices forms an interval and the relative order of these intervals corresponds to the relative order of corresponding bases.

Remark. The Ramsey property and an explicit description of the admissible ordering was given in [18]. The relational language used is however more complicated and does not preserve all automorphisms of the Fraïssé limit of $\mathcal{B}$. This makes it unsuitable for the extension property for partial automorphisms. The formulation here is a more optimized version.

The argument above together with the observation that in the Fraïssé limit **B** of $\mathcal{B}$ we have that for every finite $S \subseteq B$, $\mathrm{Cl}_{\mathbf{B}}(S) \leq 3|S|$, also gives a compact proof for the existence of an ω-categorical countable universal bowtie-free graph. This bound follows from the fact that function F^1, F^2, F^3 cannot cascade. The ω-categoricity follows from the fact that the orbit of S in $\mathrm{Aut}(\mathbf{B})$ is fully determined by the isomorphism type of $\mathrm{Cl}_{\mathbf{B}}(S)$ and there are only finitely many closures for every finite S. This, of course, is just a re-formulation of the argument in [6].

6   Concluding remarks

1.   It would be interesting to extend Theorem 1.7 to a class of structures which include non-unary functions. Perhaps this is too much to ask as EPPA is presently open even in the case of Steiner triple systems (as we remark in Section 5.2). However note that our structures involve partial functions and thus the EPPA may be easier to prove. But even for partial triple systems the EPPA seems to be presently open.

2.   On the structural Ramsey theory side open problems include Ramsey properties of finite lattices and other algebraic structures where the axioms (such as associativity) are difficult to control in an amalgamation procedure. See [19, 2] for results on Ramsey classes.

3.   The paper [19] gives a recursive construction for Ramsey classes. Is there similar result for EPPA?

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