EXISTENCE OF MODELING LIMITS FOR SEQUENCES OF SPARSE STRUCTURES

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Abstract. A sequence of graphs is FO-convergent if the probability of satisfaction of every first-order formula converges. A graph modeling is a graph, whose domain is a standard probability space, with the property that every definable set is Borel. It was known that FO-convergent sequence of graphs do not always admit a modeling limit, and it was conjectured that this is the case if the graphs in the sequence are sufficiently sparse. Precisely, two conjectures were proposed:

(1) If a FO-convergent sequence of graphs is residual, that is if for every integer d the maximum relative size of a ball of radius d in the graphs of the sequence tends to zero, then the sequence has a modeling limit.
(2) A monotone class of graphs $\mathcal{C}$ has the property that every FO-convergent sequence of graphs from $\mathcal{C}$ has a modeling limit if and only if $\mathcal{C}$ is nowhere dense, that is if and only if for each integer p there is $N(p)$ such that no graph in $\mathcal{C}$ contains the pth subdivision of a complete graph on $N(p)$ vertices as a subgraph.

In this paper we prove both conjectures. This solves some of the main problems in the area and among others provides an analytic characterization of the nowhere dense–somewhere dense dichotomy.

1. Introduction

Combinatorics is at a crossroads of several mathematical fields, including logic, algebra, probability, and analysis. Bridges have been built between these fields (notably at the instigation of Leibniz and Hilbert). From the interactions of algebra and logic is born model theory, which is founded on the duality of semantical and syntactical elements of a language. Several frameworks have been proposed to unify probability and logic, which mainly belong to two kinds: probabilities over models (Carnap, Gaifman, Scott and Kraus, Nilsson, Väänänen, Valiant,...), and models with probabilities (H. Friedman, Keisler and Hoover, Terwijn, Goldbring and Towsner,...). See [19] for a partial overview.

Recently, new bridges appeared between combinatorics and analysis, which are based on the concept of graph limits (see [21] for an in-depth exposition). Two main directions were proposed for the study of a "continuous limit" of finite graphs by means of statistics convergence:

- the *left convergence* of a sequence of (dense) graphs, for which the limit object can be either described as an *infinite exchangeable random graph*

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(that is a probability measure on the space of graphs over $\mathbb{N}$ that is invariant under the natural action of S_ω) [2, 16], or as a *graphon* (that is a measurable function $W : [0,1] \times [0,1] \to [0,1]$) [6, 7, 22].

- the *local convergence* of a sequence of bounded degree graphs, for which the limit object can be either described as a *unimodular distribution* (a probability distribution on the space of rooted connected countable graphs with bounded degrees satisfying some invariance property) [3], or as a *graphing* (a Borel graph that satisfies some Intrinsic Mass Transport Principle or, equivalently, a graph on a Borel space that is defined by means of finitely many measure preserving involutions) [9].

A general (unifying) framework has been introduced by the authors, under the generic name “structural limits” [29]. In this setting, a sequence of structures is convergent if the satisfaction probability of every formula (in a fixed fragment of first-order logic) for a (uniform independent) random assignment of vertices to the free variables converges. The limit object can be described as a probability measure on a Stone space invariant by some group action, thus generalizing approaches of [2, 16] and [3]. This may be viewed as a natural bridge between combinatorics, model theory, probability theory, and functional analysis [33].

The existence of a graphing-like limit object, called *modeling*, has been studied in [31, 35], and the authors conjectured that such a limit object exists if and only if the structures in the sequence are sufficiently “structurally sparse”. For instance, the authors conjectured that if a convergent sequence is *non-dispersive* (meaning that the structures in the sequence have no “accumulation elements”) then a modeling limit exists:

Conjecture 1 ([35]). Every convergent residual sequence of finite structures admits a modeling limit.

For the case of sequences of graphs from a monotone class (that is a class of finite graphs closed by taking subgraphs) the authors conjectured the following exact characterization, where *nowhere dense* classes [27, 28] form a large variety of classes of sparse graphs, including all classes with excluded minors (as planar graphs), bounded degree graphs and graph classes of bounded expansion [24, 25, 26].

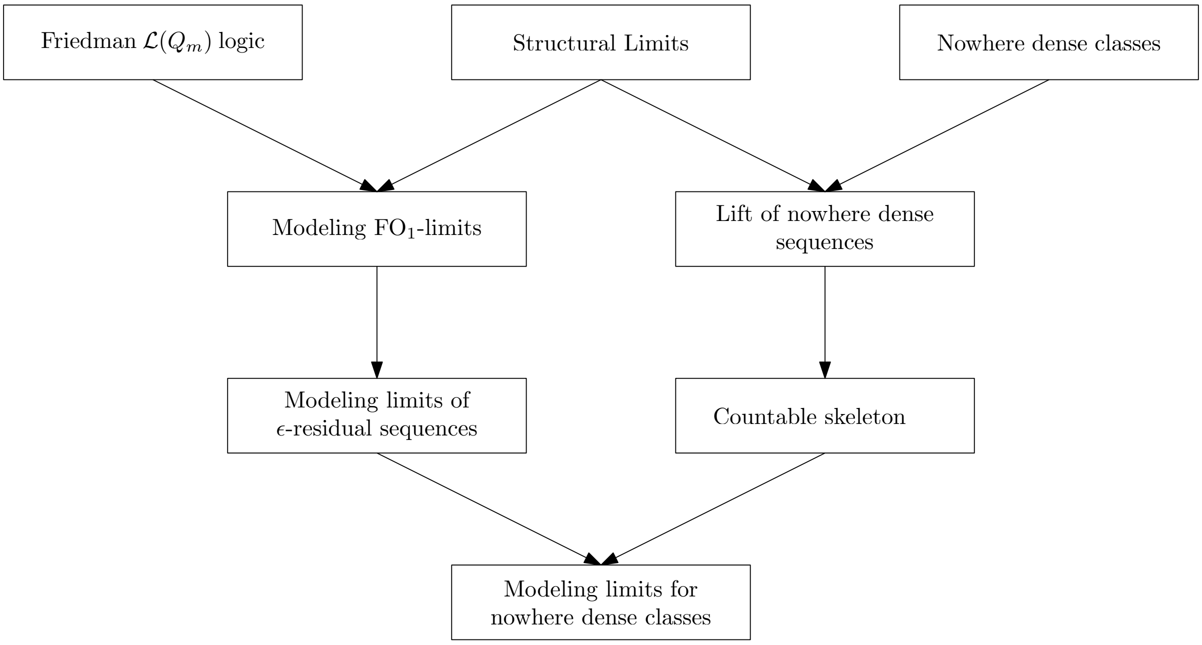
Conjecture 2 ([31]). A monotone class of graphs $\mathcal{C}$ admits modeling limits if and only if $\mathcal{C}$ is nowhere dense.

Note that this conjecture is known in one direction [31]. To prove the existence of modeling limits for sequences of graphs in a nowhere dense class is the main problem addressed in this paper.

Nowhere dense classes enjoy a number of (non obviously) equivalent characterizations and strong algorithmic and structural properties [30]. For instance, deciding properties of graphs definable in first-order logic is fixed-parameter tractable on nowhere dense graph classes (which is optimal when the considered class is monotone, under a reasonable complexity theoretic assumption) [15]. Modeling limits exist for sequences of graphs with bounded degrees (as graphings are modelings), and this has been so far verified for sequences of graphs with bounded tree-depth [31], for sequences of trees [35], for sequences of plane trees and sequences of graphs with bounded pathwidth [14], and for sequences of mappings [34] (which is the simplest form of non relational nowhere dense structures). (See also related result on sequences of matroids [17].)

In this paper, we prove both Conjecture 1 and Conjecture 2 in their full generality.

Our paper is organized as follows: In Section 2 we recall all necessary notions, definitions, and notations. In Section 3 we will deal with limits with respect to the fragment FO_1 of all first-order formulas with at most one free variable. In Section 5 we deduce a proof of Conjecture 1 and, using a characterization of nowhere denses from [36], we prove that Conjecture 2 holds. Finally, we discuss some possible developments in Section 6. The general proof strategy is depicted bellow:



2. Preliminaries, Definitions, and Notations

2.1. **Structures and Formulas.** A *signature* is a set σ of function or relation symbols, each with a finite arity. In this paper we consider finite or countable signatures. A σ*-structure* $\mathbf{A}$ is defined by its *domain* A, and by the interpretation of the symbols in σ, either as a relation $R^{\mathbf{A}}$ (for a relation symbol A) or as a function $f^{\mathbf{A}}$ (for a function symbol f). A signature σ also defines the (countable) set $\mathrm{FO}(\sigma)$ of all first-order formulas built using the relation and function symbols in σ, equality, the standard logical conjunctives, and quantification over elements of the domain. The quotient of $\mathrm{FO}(\sigma)$ by logical equivalence has a natural structure of countable Boolean algebra, the *Lindenbaum-Tarski algebra* $\mathcal{B}(\mathrm{FO}(\sigma))$ of $\mathrm{FO}(\sigma)$.

For a formula ϕ with p free variables and a structure $\mathbf{A}$ we denote by $\phi(\mathbf{A})$ the set of all satisfying assignments of ϕ in $\mathbf{A}$, that is

$$\phi(\mathbf{A}) = \{(v_1, \dots, v_p) \in A^p : \mathbf{A} \models \phi(v_1, \dots, v_p)\}.$$

If $\mathbf{A}$ is a finite structure (or a structure whose domain is a probability space), we define the *Stone pairing* $\langle \phi, \mathbf{A} \rangle$ of ϕ and $\mathbf{A}$ as the probability of satisfaction of ϕ in $\mathbf{A}$ for a random assignments of the free variables. Hence if $\mathbf{A}$ is finite (and no specific probability measure is specified on the domain of $\mathbf{A}$) it holds

$$\langle \phi, \mathbf{A} \rangle = \frac{|\phi(\mathbf{A})|}{|A|^p}.$$

Generally, if the domain of $\mathbf{A}$ is a probability space (with probability measure $\nu_{\mathbf{A}}$) and $\phi(\mathbf{A})$ is measurable then

$$\langle \phi, \mathbf{A} \rangle = \nu_{\mathbf{A}}^{\otimes p}(\phi(\mathbf{A})),$$

where $\nu_{\mathbf{A}}^{\otimes p}$ denotes the product measure on A^p.

For a σ-structure $\mathbf{A}$ we denote by Gaifman($\mathbf{A}$) the graph with vertex set A, such that two (distinct) vertices x and y are adjacent in Gaifman($\mathbf{A}$) if both belong to some relation in $\mathbf{A}$ (that is if $\exists R \in \sigma : \{x, y\} \subseteq R^{\mathbf{A}}$).

2.2. **Stone Space and Representation by Probability Measures.** The term of Stone pairing comes from a functional analysis point of view: Let $S(\mathrm{FO}(\sigma))$ be the Stone dual of the Boolean algebra $\mathcal{B}(\mathrm{FO}(\sigma))$. Points of $S(\mathrm{FO}(\sigma))$ are equivalently described as the ultrafilters on $\mathcal{B}(\mathrm{FO}(\sigma))$, the homomorphisms from $\mathcal{B}(\mathrm{FO}(\sigma))$ to the two-element Boolean algebra, or the maximal consistent sets T of formulas from $\mathrm{FO}(\sigma)$ (point of view we shall make use of here). The space $S(\mathrm{FO}(\sigma))$ is a compact totally disconnected Polish space, whose topology is generated by its clopen sets

$$k(\phi) = \{T \in S(\mathrm{FO}(\sigma)) :\ \phi \in T\}.$$

Let $\mathbf{A}$ be a finite σ-structure (or a σ-structure on a probability space such that every first-order definable set is measurable). Identifying ϕ with the indicator function $\mathbf{1}_{k(\phi)}$ of the clopen set $k(\phi)$, the map $\phi \mapsto \langle \phi, \mathbf{A} \rangle$ uniquely extends to a continuous linear form on the space $C(S(\mathrm{FO}(\sigma)))$. By Riesz representation theorem there exists a unique probability measure $\mu_{\mathbf{A}}$ such that for every $\phi \in \mathrm{FO}(\sigma)$ it holds

$$\langle \phi, \mathbf{A} \rangle = \int_{S(\mathrm{FO}(\sigma))} \mathbf{1}_{k(\phi)} \,\mathrm{d}\mu_{\mathbf{A}}.$$

Note that the permutation group S_ω defines a (subgroup of the) group of automorphisms of $\mathcal{B}(\mathrm{FO})(\sigma)$ (by permuting free variables) and acts naturally on $S(\mathrm{FO}(\sigma))$. The probability measure $\mu_{\mathbf{A}}$ associated to the structure $\mathbf{A}$ is obviously invariant under the S_ω-action.

For more details on this *representation theorem* we refer the reader to [29].

2.3. **Structural Limits.** Let σ be a signature, and let X be a fragment of $\mathrm{FO}(\sigma)$. A sequence $\mathbf{A} = (\mathbf{A}_n)_{n\in\mathbb{N}}$ of σ-structures is X*-convergent* if $\langle \phi, \mathbf{A}_n \rangle$ converges as n grows to infinity or, equivalently, if the associated probability measures $\mu_{\mathbf{A}_n}$ on $S(X)$ converge weakly [29].

In our setting, the strongest notion of convergence is FO-convergence (corresponding to the full fragment of all first-order formulas). Convergence with respect to the fragment QF^- (of all quantifier-free formulas without equality) is equivalent to the *left convergence* introduced by Lovász *et al* [4, 6, 22]. (It is also equivalent to convergence with respect to the fragment QF of all quantifier-free formulas, provided that the sizes of the structures in the sequence tend to infinity.) For bounded degree graphs, convergence with respect to the fragment $\mathrm{FO}_1^{\mathrm{local}}$ of local formulas with a single free variable is equivalent to the *local convergence* introduced by Benjamini and Schramm [3]. (Recall that a formula is *local* if its satisfaction only depends on a fixed neighborhood of its free variables.) Also, in this case, local convergence is equivalent to convergence with respect to the fragment $\mathrm{FO}^{\mathrm{local}}$ of all local formulas, provided that the sizes of the structures in the sequence tend to infinity. For a discussion on the different notions of convergence arising from different choices of the considered fragment of first-order logic, we refer the interested reader to [29, 31, 35].

Note that the equivalence of X-convergence with the weak convergence of the probability measures on $S(X)$ associated to the finite structures in the sequence is stated in [29] as a representation theorem, which generalizes both the representation of the left limit of a sequence of graphs by an infinite random exchangeable graph [2] and the representation of the local limit of a sequence of graphs with bounded degree by an unimodular distribution on the space of rooted connected countable graphs [3].

2.4. **Non-standard Limit Structures.** A construction of a non-standard limit object for FO-convergent sequences has been proposed in [29], which closely follows Elek and Szegedy construction for left limits of hypergraphs [10]. One proceeds as follows:

Let $(\mathbf{A}_n)_{n\in\mathbb{N}}$ be a sequence of finite σ-structures and let U be a non-principal ultrafilter. Let $\widetilde{A} = \prod_{i\in\mathbb{N}} A_i$ and let $\sim$ be the equivalence relation on $\widetilde{A}$ defined by $(x_n) \sim (y_n)$ if $\{n : x_n = y_n\} \in U$. Then the *ultraproduct* of the structures $\mathbf{A}_n$ is the structure $\mathbf{L} = \prod_U \mathbf{A}_i$, whose domain L is the quotient of $\widetilde{A}$ by $\sim$, and such that for each relational symbol R it holds is defined by

$$([v^1], \dots, [v^p]) \in R^{\mathbf{L}} \iff \{n : (v^1_n, \dots, v^p_n) \in R^{\mathbf{A}_n}\} \in U.$$

As proved by Łoś [20], for each formula $\phi(x_1, \dots, x_p)$ and each $v^1, \dots, v^p \in \prod_n A_n$ we have

$$\prod_U \mathbf{A}_i \models \phi([v^1], \dots, [v^p]) \quad \text{iff} \quad \{i : \ \mathbf{A}_i \models \phi(v^1_i, \dots, v^p_i)\} \in U.$$

In [29] a probability measure ν is constructed from the normalised counting measures ν_i of A_i via the Loeb measure construction, and it is proved that every first-order definable set of the ultraproduct is measurable. The ultraproduct is then a limit object for the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$. In particular, for every first-order formula ϕ with p free variables it holds:

$$\langle \phi, \prod_U \mathbf{A}_i \rangle = \int \cdots \int \mathbf{1}_\phi([x_1], \dots, [x_p])\ \mathrm{d}\nu([x_1]) \ \dots \ \mathrm{d}\nu([x_p]) = \lim_U \langle \phi, \mathbf{A}_i \rangle.$$

Moreover, the above integral is invariant by any permutation on the order of the integrations.

However, the constructed object is difficult to handle. In particular, the sigma-algebra constructed on $\prod_U \mathbf{A}_n$ is not separable. For a discussion we refer the reader to [8, 10]. The ultraproduct construction is used in the proof of Lemma 2 to prove consistency of some theories in Friedman's Q_m logic (see Section 2.6).

2.5. **Modelings.** By similarity with *graphings*, which are limit objects for local convergent sequences of graphs with bounded degrees [9], the authors proposed the term of *modeling* for a structure $\mathbf{A}$ built on a standard Borel space A, endowed with a probability measure $\nu_{\mathbf{A}}$, and such that every first-order definable set is Borel [31]. Such structures naturally avoid pathological behaviours (for instance, every definable set is either finite, countable, or has the cardinality of continuum). The definition of Stone pairing obviously extends to modeling by setting

$$\langle \phi, \mathbf{A} \rangle = \nu^{\otimes p}(\phi(\mathbf{A})). \tag{1}$$

An X-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ has *modeling X-limit* $\mathbf{L}$ (or simply *modeling limit* $\mathbf{L}$ when $X = \mathrm{FO}$) if $\mathbf{L}$ is a modeling such that for every $\phi \in X$ it holds

$$\langle \phi, \mathbf{L} \rangle = \lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle.$$

Let $\mathcal{C}$ be a class of structures. We say that $\mathcal{C}$ *admits modeling limits* if every FO-convergent sequence of structures $(\mathbf{A}_n)_{n\in\mathbb{N}}$ with $\mathbf{A}_n \in \mathcal{C}$ has a modeling limit.

Note that not every FO-convergent sequence has a modeling limit: Consider a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs, where G_n is a graph of order n, with edges drawn randomly (independently) with edge probability $0 < p < 1$. Then with probability 1 the sequence $(G_n)_{n\in\mathbb{N}}$ is FO-convergent. However, this sequence has no modeling limit, and even no modeling QF^--limit: Assume for contradiction that $(G_n)_{n\in\mathbb{N}}$ has a modeling QF-limit $\mathbf{L}$. Because $\langle x_1 = x_2, G_n \rangle = 1/n \to 0$ the probability measure $\nu_{\mathbf{L}}$ is atomless thus L is uncountable. As L is a standard Borel space, there exists zero-measure sets $N \subset L$ and $N' \subset [0,1]$, and a bijective measure preserving map $f : L \setminus N \to [0,1] \setminus N'$. By the equivalence of QF^--convergence and left-convergence the modeling $\mathbf{L}$ defines a $\{0,1\}$-valued graphon $W : [0,1] \times [0,1] \to [0,1]$, which is a left limit of $(G_n)_{n\in\mathbb{N}}$ by:

$$W(x,y) = \begin{cases} 1 & \text{if } x, y \notin N' \text{ and } \mathbf{L} \models f^{-1}(x) \sim f^{-1}(y) \\ 0 & \text{otherwise.} \end{cases}$$

But a left limit of $(G_n)_{n\in\mathbb{N}}$ is the constant graphon p, which is not weakly equivalent to W (as it should, according to [5]) thus we are led to a contradiction.

This example is prototypal, and this allows us to prove that if a monotone class of graphs admits modeling limits then this class has to be nowhere dense [31]. The proof involves the characterization of nowhere dense classes by the model theoretical notions of stability and independence property [1], their relation to VC-dimension [18], and the characterization of sequences of graphs admitting a *random-free* (i.e. almost everywhere $\{0,1\}$-valued) left limit graphon [23]. Conjecture 2 asserts that the converse is true as well.

2.6. **H. Friedman's Q_m-logic.** Friedman [11, 12] studied a logical system where the language is enriched by the quantifier "there exists x in a non zero-measure set ...", for which he studied axiomatizations, completeness, decidability, etc. A survey including all these results was written by Steinhorn [37, 38]. In particular, H. Friedman considered specific type of models, which he calls *totally Borel*, which are (almost) equivalent to our notion of modeling: A *totally Borel structure* is a structure whose domain is a standard Borel space (endowed with implicit Borel measure) with the property that every first-order definable set (with parameters) is measurable.

In this context, Friedman introduced a new quantifier Q_m, which is to be understood as expressing "there exists non-measure 0 many", and initiated the study of the extension $\mathcal{L}(Q_m)$ of first order logic, whose axioms are all the usual axiom schema for first-order logic together with the following ones [38]:

M_0 $\neg(Q_m x)(x = y)$;

M$_1$ $(Q_m x)\Psi(x,\dots) \leftrightarrow (Q_m y)\Psi(y,\dots)$, where $\Psi(x,\dots)$ is an $\mathcal{L}(Q_m)$-formula in which y does not occur and $\Psi(y,\dots)$ is the result of replacing each free occurrence of x by y;

M$_2$ $(Q_m x)(\Phi \vee \Psi) \to (Q_m x)\Phi \vee (Q_m x)\Psi$;

M$_3$ $[(Q_m x)\Phi \wedge (\forall x)(\Phi \to \Psi)] \to (Q_m x)\Psi$;

M$_4$ $(Q_m x)(Q_m y)\Phi \to (Q_m y)(Q_m x)\Phi$.

The rules of inference for $\mathcal{L}(Q_m)$ are the same as for first-order logic: *modus ponens* and generalization. Let the proof system just described be denoted by $\mathcal{K}_m$.

The standard semantic for Q_m is as follows: for a structure $\mathbf{M}$ on a probability space such that every first-order definable (with parameters) is measurable (for probability measure λ) it holds

$$\mathbf{M} \models Q_m x\ \phi(x,\overline{a}) \iff \lambda(\{x : \mathbf{M} \models \phi(x,\overline{a})\}) > 0.$$

Note that the set of $\mathcal{L}(Q_m)$-sentences satisfied by $\mathbf{M}$ (for this semantic) is obviously consistent in $\mathcal{K}_m$.

The following completeness theorem has been proved by Friedman [11] (see also [38]):

Theorem 1. *A set of sentences T in $\mathcal{L}(Q_m)$ has a totally Borel model if and only if T is consistent in $\mathcal{K}_m$.*

It has been noted that one can require the domain of the totally Borel model to be a Borel subset of $\mathbb{R}$ with Lebesgue measure 1.

3. Modeling FO$_1$-limits

Let $\mathsf{A} = (\mathbf{A}_n)_{n\in\mathbb{N}}$ be an FO-convergent sequence of finite structures, and let $T(\mathsf{A})$ be the union of a complete theory of an elementary limit of A together with, for each first order formula ϕ with free variables $x_1,\dots,x_p$,

$$\begin{array}{lll} \text{either} & (Q_m x_1)\dots(Q_m x_p)\ \phi, & \text{if } \lim_{n\to\infty} \langle \phi, \mathbf{A}_n\rangle > 0; \\ \text{or} & \neg\big((Q_m x_1)\dots(Q_m x_p)\ \phi\big), & \text{if } \lim_{n\to\infty} \langle \phi, \mathbf{A}_n\rangle = 0. \end{array}$$

The ultraproduct construction provides a model for $T(\mathsf{A})$:

Lemma 2. *For every FO-convergent sequence A of finite structures, the theory $T(\mathsf{A})$ is consistent in $\mathcal{K}_m$.*

Proof. Using the standard semantic for Q_m it is immediate that any ultraproduct $\prod_U \mathbf{A}_i$ is a model for $T(\mathsf{A})$ hence $T(\mathsf{A})$ is consistent in $\mathcal{K}_m$. □

Theorem 3. *For every FO-convergent sequence A of finite structures, there exists a modeling $\mathbf{M}$ whose domain M is a Borel subset of $\mathbb{R}$, and such that:*

(1) the probability measure $\nu_{\mathbf{M}}$ associated to $\mathbf{M}$ is uniformly continuous with respect to Lebesgue measure λ;

(2) $\mathbf{M}$ is a modeling FO$_1$-limit of A;

(3) for every $\phi \in$ FO it holds

$$\langle \phi, \mathbf{M}\rangle = 0 \iff \lim_{n\to\infty} \langle \phi, \mathbf{A}_n\rangle = 0.$$

Proof. According to Lemma 2 the theory $T(\mathbf{A})$ is consistent in $\mathcal{K}_m$. Hence, according to Theorem 1, $T(\mathbf{A})$ has a totally Borel model $\mathbf{T}$. (Furthermore, we may assume that T is a Borel subset of $\mathbb{R}$ with Lebesgue measure 1.)

For every integer k, there exists an integer $N(k)$ and $N(k)$ formulas $\theta_1^k, \dots, \theta_{N(k)}^k$ (with a single free variable) defining the local 1-types up to quantifier rank k in the sense that all of these formulas are local and have quantifier rank k, they induce a partition (formalized as $\theta_i^k \vdash \neg\theta_j^k$ if $i \neq j$ and $\vdash \bigvee_i \theta_i^k$), and for every local formula $\phi(x)$ with quantifier rank k and for every $1 \leq i \leq N(k)$ either it holds $\theta_i^k \vdash \phi$, or $\theta_i^k \vdash \neg\phi$.

Define $I_k = \{i : \lambda(\theta_i^k(\mathbf{L})) > 0\}$. Define the probability measure π_k on L as follows: for every Borel subset X of L define

$$\pi_k(X) = \sum_{i \in I_k} \frac{\lambda(X \cap \theta_i^k(\mathbf{L}))}{\lambda(\theta_i^k(\mathbf{L}))} \cdot \lim_{n\to\infty} \langle \theta_i^k, G_n \rangle.$$

Obviously π_k weakly converges to some probability measure π. Let $\mathbf{M}$ be the modeling obtained by endowing $\mathbf{L}$ with the probability measure $\nu_{\mathbf{M}} = \pi$. Note that $\nu_{\mathbf{M}}$ is absolutely continuous with respect to λ by construction. □

Theorem 3 immediately implies

Corollary 1. *Every* FO_1*-convergent sequence has a modeling* FO_1*-limit.*

4. Modeling Limits of Residual Sequences

We know that in general an FO-convergent sequence does not have a modeling limit (hence Corollary 1 does not extend to full FO). This nicely relates to sparse–dense dichotomy.

Recall that a class $\mathcal{C}$ of (finite) graphs is *nowhere dense* if, for every integer k, there exists an integer $n(k)$ such that the k-th subdivision of the complete graph $K_{n(k)}$ on $n(k)$ vertices is the subgraph of no graph in $\mathcal{C}$ [27, 30]. (Note a subgraph needs not to be induced.) Based on a characterization by Lovász and Szegedy [23] or random-free graphon and a characterization of nowhere-dense classes in terms of VC-dimension (Adler and Adler [1] and Laskowski [18]) the authors derived in [31] the following necessary condition for a monotone class $\mathcal{C}$ to have modeling limits.

Theorem 4. *Let $\mathcal{C}$ be a monotone class of graphs. If every FO-convergent of graphs from $\mathcal{C}$ has a modeling limit then the class $\mathcal{C}$ is nowhere dense.*

However, there is a particular case where a modeling limit for an FO-convergent sequence will easily follow from Theorem 3. That will be done next.

Definition 5. A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is *residual* if, for every integer d it holds

$$\lim_{n\to\infty} \sup_{v_n \in A_n} \frac{|B_d(\mathbf{A}_n, v_n)|}{|A_n|} = 0,$$

where $B_d(\mathbf{A}_n, v_n)$ denotes the set of elements of A_n at distance at most d from v_n (in the Gaifman graph of $\mathbf{A}_n$). Equivalently, $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is residual if, for every integer d, it holds

$$\lim_{n\to\infty} \langle \mathrm{dist}(x_1, x_2) \leq d, \mathbf{A}_n \rangle = 0.$$

The notion of residual sequence is linked to the one of residual modeling: A *residual modeling* is a modeling, all components of which have zero measure (that is if and only if for every integer d, every ball of radius d has zero measure).

By an interplay of these notions we now can prove Conjecture 1.

Theorem 6. *Every* FO*-convergent residual sequence has a modeling limit.*

Proof. The main characteristic of residual sequences is that a residual sequence is FO-convergent if and only if it is FO_1-convergent [35]. Consider the modeling limit $\mathbf{M}$ obtained in Theorem 3 for a FO-convergent residual sequence. Then for every integer d it holds

$$\langle \mathrm{dist}(x_1, x_2) \leq d, \mathbf{M}\rangle = 0.$$

It follows that $\mathbf{M}$ is residual, and thus the convergence of $\langle \phi, \mathbf{A}_n\rangle$ to $\langle \phi, \mathbf{M}\rangle$ for first-order formulas with (at most) one free variable (i.e. FO_1-convergence) implies convergence for all first-order formulas (i.e. FO-convergence). □

5. Modeling Limits of Quasi-Residual Sequences

Here we prove our main result in the form of a generalization of Section 4 for quasi-residual sequences. The motivation for the introduction of the definition of quasi-residual sequences is the following:

Known constructions of modeling limits for some nowhere dense classes with unbounded degrees [14, 31, 35] are based on the construction of a countable "skeleton" on which residual parts are grafted. We shall use the same idea here for the general case. The countable skeleton will be built thanks to the following characterization of nowhere dense classes proved in [36]:

Theorem 7. *Let $\mathcal{C}$ be a class of graphs. Then $\mathcal{C}$ is nowhere dense if and only if for every integer d and every $\epsilon > 0$ there is an integer $N = N(d, \epsilon)$ with the following property: for every graph $G \in \mathcal{C}$, and every subset A of vertices of G, there is $S \subseteq A$ with $|S| \leq N$ such that no ball of radius d in $G[A \setminus S]$ has order greater than $\epsilon\,|A|$.*

This theorem justifies the introduction of the following relaxation of the notion of residual sequence:

Definition 8. A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ (with $|A_n| \to \infty$) is *quasi-residual* if, for every integer d and every $\epsilon > 0$ there exists an integer N such that it holds

$$\limsup_{n\to\infty} \inf_{S_n \in \binom{A_n}{N}} \sup_{v_n \in A_n \setminus S_n} \frac{|B_d(\mathrm{Gaifman}(\mathbf{A}_n) \setminus S_n, v_n)|}{|A_n|} < \epsilon.$$

In other words, $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is quasi-residual if, for every distance d and every $\epsilon > 0$ there exists an integer N so that (for sufficiently large n) one can remove at most N vertices in the Gaifman graph of $\mathbf{A}_n$ so that no ball of radius d will contain at least ϵ proportion of A_n.

The next result directly follows from Theorem 7.

Corollary 2. *Let $\mathcal{C}$ be a nowhere dense class of graphs and let $(G_n)_{n\in\mathbb{N}}$ be a sequences of graphs from $\mathcal{C}$ such that $|G_n| \to \infty$. Then $(G_n)_{n\in\mathbb{N}}$ is quasi-residual.*

5.1. (d,ϵ)**-residual Sequences.** We now consider a relaxation of the notion of residual sequence and show how this allows to partially reduce the problem of finding modeling FO-limits to finding modeling FO_1-limits.

Definition 9. Let d be an integer and let ϵ be a positive real. A sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is (d,ϵ)-*residual* if it holds

$$\limsup_{n\to\infty} \sup_{v_n\in A_n} \frac{|B_d(\mathbf{A}_n, v_n)|}{|A_n|} < \epsilon.$$

Similarly, a modeling $\mathbf{M}$ is (d,ϵ)-*residual* if it holds

$$\sup_{v\in M} \nu_{\mathbf{M}}(B_d(\mathbf{M}, v)) < \epsilon.$$

Lemma 10. *Let $d\in\mathbb{N}$ and let $\epsilon>0$ be a positive real. Assume $(\mathbf{A}_n)_{n\in\mathbb{N}}$ is a* FO*-convergent $(2d,\epsilon)$-residual sequence of graphs and assume $\mathbf{L}$ is a $(2d,\epsilon)$-residual modeling* FO_1*-limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$.*

Then for every d-local formula ϕ with p free variables it holds

$$|\langle\phi,\mathbf{L}\rangle - \lim_{n\to\infty}\langle\phi,\mathbf{A}_n\rangle| < p^2\epsilon.$$

Proof. By restricting the signature to the symbols in ϕ if necessary, we can assume that σ is finite. Let q be the quantifier rank of ϕ. Then there exists finitely many local formula $\xi_1,\dots,\xi_N$ with quantifier rank at most q (expressing the rank q d-local type) such that:

- every element of every model satisfies exactly one of the ξ_i (formally, $\vdash \bigvee \xi_i$ and $\vdash (\xi_i \to \neg\xi_j)$ if $i\neq j$);
- two elements x and y satisfies the same local first-order formulas of quantifier rank at most q if and only if they satisfy the same ξ_i.

Let $\zeta(x_1,\dots,x_p)$ be the formula $\bigwedge_{1\leq i<j\leq p} \mathrm{d}_{>2d}(x_i,x_j)$. By d-locality of ϕ there exists a subset $\mathcal{X}\subseteq [N]^p$ such that

$$\zeta \vdash \Big[\phi \leftrightarrow \bigvee_{(i_1,\dots,i_p)\in\mathcal{X}} \bigwedge_{j=1}^{p} \xi_{i_j}(x_j)\Big].$$

Let $\widetilde{\phi} = \bigvee_{(i_1,\dots,i_p)\in\mathcal{X}} \bigwedge_{j=1}^{p} \xi_{i_j}(x_j)$. For every structure $\mathbf{A}$ it holds

$$\langle\widetilde{\phi},\mathbf{A}\rangle = \sum_{(i_1,\dots,i_p)\in\mathcal{X}} \prod_{j=1}^{p} \langle\xi_{i_j},\mathbf{A}\rangle.$$

As $\mathbf{L}$ is a modeling FO_1-limit of $\mathbf{A}_n$ it holds $\langle\xi_{i_j},\mathbf{L}\rangle = \lim_{n\to\infty}\langle\xi_{i_j},\mathbf{A}_n\rangle$, hence

$$\langle\widetilde{\phi},\mathbf{L}\rangle = \lim_{n\to\infty}\langle\widetilde{\phi},\mathbf{A}_n\rangle.$$

On the other hand, as $\zeta\vdash(\phi\leftrightarrow\widetilde{\phi})$, for every structure $\mathbf{A}$ holds

$$|\langle\phi,\mathbf{A}\rangle - \langle\widetilde{\phi},\mathbf{A}\rangle| \leq \langle\neg\zeta,\mathbf{A}\rangle \leq \binom{p}{2}\langle \mathrm{d}_{\leq 2d},\mathbf{A}\rangle.$$

Note that $\langle \mathrm{d}_{\leq 2d},\mathbf{A}\rangle$ is nothing but the expected measure of a ball of radius $2d$ in $\mathbf{A}$. In particular, if $\mathbf{A}$ is $(2d,\epsilon)$-residual, then it holds $|\langle\phi,\mathbf{A}\rangle - \langle\widetilde{\phi},\mathbf{A}\rangle| < \epsilon$. Thus,

$$|\langle\phi,\mathbf{L}\rangle - \lim_{n\to\infty}\langle\phi,\mathbf{A}_n\rangle| < p^2\epsilon.$$

□

5.2. **Marked Quasi-residual sequences.** To allow an effective use of the properties of quasi-residual sequences, we use a (lifted) variant of the notion of quasi-residual sequence.

Let σ be a countable signature and let σ^+ be the signature obtained by adding to σ countably many unary symbols $\{M_i\}_{i\in\mathbb{N}}$ and $\{Z_i\}_{i\in\mathbb{N}}$.

For integers d, i we define the formulas $\delta_{d,i}$ and $\hat{\delta}_d$ as

$$\delta_{d,i} := (\exists z)\ \mathrm{d}_{\leq d}(x_1, z) \wedge M_i(z) \tag{2}$$

$$\hat{\delta}_d := (\exists z)\ \mathrm{d}_{\leq d}(x_1, z) \wedge Z_d(z) \tag{3}$$

In other words, $\delta_{d,i}(x)$ holds if x belongs to the ball of radius d centered at the element marked M_i, and $\hat{\delta}_d(x)$ holds if x belongs to the d-neighborhood of elements marked by Z_d.

Definition 11. A sequence $(\mathbf{A}_n^+)_{n\in\mathbb{N}}$ (with $|A_n^+| \to \infty$) of σ^+-structures is a *marked quasi-residual* sequence if the following condition holds:

- For every integers i, n it holds $|M_i(\mathbf{A}_n^+)| \leq 1$ (i.e. at most one element in $\mathbf{A}_n^+$ is marked by M_i);
- For every distinct integers i, j and every integer n, no element of $\mathbf{A}_n^+$ is marked both M_i and M_j;
- For every integer d there is a non-decreasing unbounded function $F_d : \mathbb{N} \to \mathbb{N}$ with the property that for every integer n it holds

$$Z_d(\mathbf{A}_n^+) = \bigcup_{i=1}^{F_d(n)} M_i(\mathbf{A}_n^+); \tag{4}$$

- For every integer d and every positive real $\epsilon > 0$ there is $N \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \sup_{v_n \in A_n^+ \setminus \bigcup_{i=1}^N M_i(\mathbf{A}_n^+)} \frac{|B_d(\mathrm{Gaifman}(\mathbf{A}_n^+) \setminus \bigcup_{i=1}^N M_i(\mathbf{A}_n^+), v_n)|}{|A_n^+|} < \epsilon. \tag{5}$$

  (In other words, every ball of radius d in $\mathrm{Gaifman}(\mathbf{A}_n^+) \setminus \bigcup_{i=1}^N M_i(\mathbf{A}_n^+)$ contains less than ϵ proportion of all the vertices, as soon as n is sufficiently large.)
- For every integer d the following limit equality holds:

$$\lim_{n\to\infty} \langle \hat{\delta}_d, \mathbf{A}_n^+ \rangle = \lim_{m\to\infty} \lim_{n\to\infty} \langle \bigwedge_{i=1}^{m} \delta_{d,m}, \mathbf{A}_n^+ \rangle. \tag{6}$$

The main purpose of this admittedly technical definition is to allow to make use of the sets S_n arising in the definition of quasi-residual sequences by first-order formula, by means of the marks M_i. The role of the marks Z_d is to allow a kind of of limit exchange. (Note that $\delta_{d,i}(\mathbf{A}^+)$ is nothing but the ball of radius d of $\mathbf{A}^+$ centered at the element marked by M_i.)

Lemma 12. *For every quasi-residual sequence* $(\mathbf{A}_n)_{n\in\mathbb{N}}$ *of* σ*-structures there exists an* FO*-convergent marked quasi-residual sequence* $(\mathbf{B}_n^+)_{n\in\mathbb{N}}$ *of* σ*-structures such that* $(\mathrm{Forget}(\mathbf{B}_n^+))_{n\in\mathbb{N}}$ *is a subsequence of* $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Proof. Let σ' be the signature obtained by adding to σ countably many unary symbols $\{M_i\}_{i\in\mathbb{N}}$. For $n \in \mathbb{N}$ we define the σ'-structure $\mathbf{A}'_n$ has the σ'-structure obtained from $\mathbf{A}_n$ by defining marks M_i are assigned in such a way that for every $d \in \mathbb{N}$ and $\epsilon > 0$ there is $N \in \mathbb{N}$ such that letting $S_n = \bigcup_{i=1}^N M_i(\mathbf{A}'_n)$ it holds

$$\limsup_{n\to\infty} \sup_{v_n \in A'_n \setminus S_n} \frac{|B_d(\mathrm{Gaifman}(\mathbf{A}'_n) \setminus S_n, v_n)|}{|A'_n|} < \epsilon.$$

This is obviously possible, thanks to the definition of a quasi-residual sequence.

Considering an FO-convergent subsequence we may assume that $(\mathbf{A}'_n)$ is FO-convergent.

For $d \in \mathbb{N}$ we define the constant

$$\alpha_d = \lim_{m\to\infty} \lim_{n\to\infty} \langle \bigvee_{i=1}^m \delta_{d,i}, \mathbf{A}'_n \rangle.$$

(Note that the values $\lim_{n\to\infty}\langle \bigvee_{i=1}^m \delta_{d,i}, \mathbf{A}'_n\rangle$ exist as $(\mathbf{A}'_n)$ is FO-convergent and that they form, for increasing m, a non-decreasing sequence bounded by 1.)

Then for each $d \in \mathbb{N}$ there exists a non-decreasing function $F_d : \mathbb{N} \to \mathbb{N}$ such that $\lim_{n\to\infty} \lim F_d(n) = \infty$ and

$$\lim_{n\to\infty} \langle \bigvee_{i=1}^{F(n)} \delta_{d,i}, \mathbf{A}'_n \rangle = \alpha_d.$$

Then we define $\mathbf{A}^+_n$ to be the sequence obtained from $\mathbf{A}'_n$ by marking by Z_d all the elements in $\bigcup_{i=1}^{F_d(n)} M_i(\mathbf{A}'_n)$. Now we let $(\mathbf{B}^+_n)$ to be a converging subsequence of $(\mathbf{A}^+_n)_{n\in\mathbb{N}}$. □

Let ζ_d be the formula asserting that the ball of radius d centered at x_1 contains x_2 but no element marked Z_d, that is

$$\zeta_d := \mathrm{d}_{\leq d}(x_1, x_2) \wedge (\forall z)(\mathrm{d}_{\leq d}(x_1, z) \to \neg Z_d(z)).$$

Lemma 13. *Let $(\mathbf{A}^+_n)_{n\in\mathbb{N}}$ be a marked quasi-residual sequence. Then*

$$\lim_{n\to\infty} \langle \zeta_d, \mathbf{A}^+_n \rangle = 0.$$

Proof. Assume for contradiction that $a = \lim_{n\to\infty}\langle \zeta_d, \mathbf{A}^+_n\rangle$ is strictly positive.

According to the definition of a marked quasi-residual sequence, there exists an integer m such that no ball of radius d in $\mathrm{Gaifman}(\mathbf{A}^+_n) \setminus \bigcup_{i=1}^m M_i(\mathbf{A}^+_n)$ contains more than $(a/2)|A_n|$ elements. Let n_0 be such that $F_d(n_0) \geq m$, and let $n_1 \geq n_0$ be such that $\langle \zeta_d, \mathbf{A}^+_n\rangle > a/2$ holds for every $n \geq n_1$.

Then there exists v such that the ball of radius d centered at v contains no element marked Z_d (hence no element marked $M_1, \ldots, M_m$) and contains more than $(a/2)|A_n|$ elements, what contradicts the fact that this ball is a ball of radius d in $\mathrm{Gaifman}(\mathbf{A}^+_n) \setminus \bigcup_{i=1}^m M_i(\mathbf{A}^+_n)$. □

In general, a modeling FO_1-limit of a (d, ϵ)-residual sequence does not need to be (d', ϵ')-residual. However, if we consider a sequence that is also marked quasi-residual, and if we assume that the modeling FO_1-limit satisfies the additional properties asserted by Theorem 3 then we can conclude that the modeling is $(d/4, \epsilon)$-residual, as proved in the next lemma.

Lemma 14. *If the sequence $(\mathbf{A}^+_n)$ is $(4d, \epsilon)$-residual and $\mathbf{L}^+$ is a modeling with the properties asserted by Theorem 3 then $\mathbf{L}^+$ is (d, ϵ)-residual.*

Proof. We first prove that the set Υ of vertices $v \in L^+$ such that the ball of radius $2d$ centered at v has measure greater than ϵ has zero measure. According to Lemma 13, it holds $\lim_{n\to\infty}\langle \zeta_{2d}, \mathbf{A}_n^+\rangle = 0$ hence $\langle \zeta_{2d}, \mathbf{L}^+\rangle = 0$. This implies that the set V of x_1 such that the ball of radius $2d$ centered at x_1 contains no element marked Z_{2d} and has measure at least ϵ has zero measure. Hence we only have to consider vertices v in the $2d$-neighborhood of $Z_{2d}(\mathbf{L}^+)$. Let

$$\alpha_{2d} = \lim_{m\to\infty}\lim_{n\to\infty}\langle \bigvee_{i=1}^{m} \delta_{2d,i}, \mathbf{A}_n+\rangle.$$

Let $k \in \mathbb{N}$. There exists $m(k)$ such that

$$\lim_{n\to\infty}\langle \bigvee_{i=1}^{m(k)} \delta_{2d,i}, \mathbf{A}_n^+\rangle > \alpha_{2d} - 1/k, \tag{7}$$

which means that at least $\alpha_{2d} - 1/k$ proportion of $\mathbf{L}^+$ is at distance at most $2d$ from elements marked $M_1, \dots, M_{m(k)}$.

However, according to (6), and as $\mathbf{L}^+$ is a modeling FO_1-limit of $(\mathbf{A}_n^+)_{n\in\mathbb{N}}$ it holds

$$\alpha_{2d} = \lim_{n\to\infty}\langle \hat{\delta}_{2d}, \mathbf{A}_n^+\rangle = \langle \hat{\delta}_{2d}, \mathbf{L}^+\rangle,$$

which means that a α_{2d} proportion of $\mathbf{L}^+$ is at distance at most $2d$ from elements marked Z_{2d} (which include elements marked $M_1, \dots, M_{m(k)}$). Thus the set N_k of vertices in the $2d$-neighborhood of $Z_{2d}(\mathbf{L}^+)$ but not in the $2d$-neighborhood of $\bigcup_{i=1}^{m(k)} M_i(\mathbf{L}^+)$ has measure at most $1/k$.

Let v be in the $2d$-neighborhood of $\bigcup_{i=1}^{m(k)} M_i(\mathbf{L}^+)$. Then the ball of radius $2d$ centered at v is included in the ball of radius $4d$ centered at a vertex marked M_i, for some $i \le m(k)$. But this ball has measure $\langle \delta_{4d,i}, \mathbf{L}^+\rangle = \lim_{n\to\infty}\langle \delta_{4d,i}, \mathbf{A}_n^+\rangle$. As the sequence $(\mathbf{A}_n^+)$ is $(4d, \epsilon)$-residual, it holds $\langle \delta_{4d,i}, \mathbf{A}_n^+\rangle < \epsilon$ for sufficiently large n. Hence the ball of $\mathbf{L}^+$ of radius $2d$ centered at v (which is included in the ball of radius $4d$ centered at the vertex marked M_i) has measure less than ϵ.

It follows that the set of v such that the ball of radius $2d$ centered at v has measure at least ϵ is included in $V \cup \bigcap_k N_k$ hence has zero measure.

Now assume for contradiction that there exists a vertex v such that the ball B of radius d centered at v has measure at least ϵ. Then for every $w \in B$ the ball of radius $2d$ centered at v has measure at least ϵ, what contradicts the fact that the measure of B is positive. □

5.3. **Color Coding and Mark Elimination.** We now consider how to turn a marked quasi-residual into a (d, ϵ)-residual marked quasi-residual sequence.

The idea here, is to encode each relation R with arity $k > 1$ with $m^k - 1$ relations plus a sentence. The sentence expresses the behaviour of R when restricted to elements marked $M_1, \dots, M_m$. The $m^k - 1$ relations expresses which tuples of non-marked elements can be extended (and how) with elements marked $M_1, \dots, M_m$ to form a k-tuple of R.

As above, let σ^+ be a countable signature with unary relations M_i and Z_i. Let $m \in \mathbb{N}$.

We define the signature σ^{*m} as the signature obtained from σ^+ by adding, for each symbol $R \in \sigma$ with arity $k > 1$ the relation symbols $N_{I,f}^R$ of arity $k - |I|$, where $\emptyset \neq I \subsetneq [k]$ and $f : I \to [m]$.

Let $\mathbf{A}^+$ be a σ^+-structure.

We define the structure $\mathrm{Encode}_m(\mathbf{A}^+)$ as the σ^{*m}-structure $\mathbf{A}^*$, which has same domain as $\mathbf{A}^+$, same unary relations, and such that for every symbol $R \in \sigma^+$ with arity $k > 1$, for every $\emptyset \neq I \subsetneq [k]$ and $f : I \to [m]$, denoting $i_1 < \cdots < i_\ell$ the elements of $[k] \setminus I$ and $i_{\ell+1}, \ldots, i_k$ the elements of I, it holds

$$\begin{aligned}&\mathbf{A}^* \models N^R_{I,f}(v_{i_1}, \ldots, v_{i_\ell})\\ &\iff \mathbf{A}^+ \models \bigwedge_{j=1}^{\ell} \bigwedge_{r=1}^{m} \neg M_r(v_{i_j}) \wedge \Big[(\exists v_{i_{\ell+1}}, \ldots, v_{i_k})\big(R(v_1, \ldots, v_k) \wedge \bigwedge_{j=\ell+1}^{k} M_{f(i_j)}(v_{i_j})\big)\Big]\end{aligned}$$

and

$$\begin{aligned}&\mathbf{A}^* \models R(v_1, \ldots, v_k)\\ &\iff \mathbf{A}^+ \models R(v_1, \ldots, v_k) \wedge \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{m} \neg M_j(v_i).\end{aligned}$$

Note that the Gaifman graph of $\mathbf{A}^*$ can be obtained from the Gaifman graph of $\mathbf{A}^+$ by removing all edges incident to a vertex marked $M_1, \ldots, M_m$.

We now explicit how the relation R in $\mathbf{A}^+$ can be retrieved from $\mathbf{A}^*$.

For $m \in \mathbb{N}$, $R \in \sigma$ with arity $k > 1$, and $\mathcal{Z} \subseteq [m]^k$ let $\eta_R^{\mathcal{Z},m}(x_1, \ldots, x_k)$ be defined as follows:

$$\begin{aligned}\eta_R^{\mathcal{Z},m} := &\bigvee_{(i_1,\ldots,i_k)\in\mathcal{Z}} \bigwedge_{j=1}^{k} M_{i_j}(x_i) \vee \Big[R(x_1, \ldots, x_k) \wedge \bigwedge_{i=1}^{k} \bigwedge_{j=1}^{m} \neg M_j(x_i)\Big]\\ &\vee \bigvee_{\emptyset\neq I\subseteq[k]} \bigvee_{f:I\to[m]} \Big[N_{I,f}(x_{i_1}, \ldots, x_{i_\ell}) \wedge \bigwedge_{i\in I} M_{f(i)}(x_i) \wedge \bigwedge_{i\in[k]\setminus II} \bigwedge_{j=1}^{m} \neg M_j(x_i)\Big]\end{aligned}$$

and let $\varsigma_R^{\mathcal{Z}}$ be the following sentence, which expresses that $\mathcal{Z}$ encodes the set of all the tuples of elements marked $M_1, \ldots, M_m$ in R.

$$\begin{aligned}\varsigma_R^{\mathcal{Z}} := \Big[&\bigwedge_{(i_1,\ldots,i_k)\in\mathcal{Z}} (\exists x_1, \ldots, x_k)\big(R(x_1, \ldots, x_k) \wedge \bigwedge_{j=1}^{k} (M_{i_j}(x_i)\big)\Big]\\ &\wedge \neg\Big[\bigvee_{(i_1,\ldots,i_k)\in[m]^k\setminus\mathcal{Z}} (\exists x_1, \ldots, x_k)\big(R(x_1, \ldots, x_k) \wedge \bigwedge_{j=1}^{k} (M_{i_j}(x_i)\big)\Big].\end{aligned}$$

The following lemma sums up the main properties of our construction.

Lemma 15. *Let $\mathbf{A}^+$ be a σ^+-structure, and let $\mathbf{A}^* = \mathrm{Encode}_m(\mathbf{A}^+)$.*

Let $R \in \sigma$ be a relation symbol with arity $k > 1$. Then

- *there exists a unique subset $\mathcal{Z}$ of $[m]^k$ such that $\mathbf{A}^+ \models \varsigma_R^{\mathcal{Z}}$*
- *for this $\mathcal{Z}$ and for every $v_1, \ldots, v_k \in A^+$ it holds*

$$\mathbf{A}^+ \models R(v_1, \ldots, v_k) \quad \iff \quad \mathbf{A}^* \models \eta_R^{\mathcal{Z},m}(v_1, \ldots, v_k).$$

Proof. This lemma straightforwardly follows from the above definitions. □

Let $m \in \mathbb{N}$ be fixed.

An *elimination theory* is a set T_m containing, for each $R \in \sigma$ with arity $k > 1$, exactly one sentence $\varsigma_R^{\mathcal{Z}}$ (for some $\mathcal{Z} \subseteq [m]^k$). For a σ^+-structure $\mathbf{A}^+$, the *elimination theory of* $\mathbf{A}^+$ is the set of all sentences $\varsigma_R^{\mathcal{Z}}$ satisfied by $\mathbf{A}^+$.

For a formula $\phi \in \mathrm{FO}(\sigma)$, we define the *elimination formula* $\widehat{\phi}$ of ϕ with respect to an elimination theory T_m as the formula obtained from ϕ by replacing each occurence of relation symbol R with arity $k > 1$ by the formula $\eta_R^{\mathcal{Z},m}$, where $\mathcal{Z}$ is the unique subset of $[m]^k$ such that $\varsigma_R^{\mathcal{Z}} \in T_m$.

It directly follows from Lemma 15 that if $\mathbf{A}^+$ is a σ^+-structure which satisfies all sentences in an elimination theory T_m, then for every formula $\phi \in \mathrm{FO}(\sigma)$, denoting $\widehat{\phi}$ the elimination formula of ϕ with respect to T_m it holds

$$\mathrm{Encode}_m(\mathbf{A}^+) \models \widehat{\phi}(v_1, \ldots, v_p) \quad \iff \quad \mathbf{A}^+ \models \phi(v_1, \ldots, v_p). \tag{8}$$

5.4. **Modeling Limits of Quasi-residual Sequences.** Let us recall Gaifman locality theorem.

Theorem 16 ([13])**.** *Every first-order formula $\psi(x_1, \ldots, x_n)$ is equivalent to a Boolean combination of t-local formulae $\chi(x_{i_1}, \ldots, x_{i_s})$ and basic local sentences of the form*

$$\exists y_1 \ldots y_m \left(\bigwedge_{i=1}^{m} \phi(y_i) \ \wedge \bigwedge_{1 \leq i < j \leq m} \mathrm{d}_{>2r}(y_i, y_j) \right)$$

where ϕ is r-local. Furthermore $r \leq 7^{\mathrm{qr}(\psi)-1}$, $t \leq 7^{\mathrm{qr}(\psi)-1}/2$, $m \leq n + \mathrm{qr}(\psi)$, and, if ψ is a sentence, only basic local sentences occur in the Boolean combination.

From this theorem we deduce:

Lemma 17. *Let $(\mathbf{A}_n)_{n \in \mathbb{N}}$ be an elementary convergent sequence of σ-structures. Then for every formula $\phi \in \mathrm{FO}(\sigma)$ with quantifier rank q there exists a $7^{q-1}/2$-local formula $\widetilde{\phi}$ and an integer n_0 such that for every $n \geq n_0$ it holds $\phi(\mathbf{A}_n) = \widetilde{\phi}(\mathbf{A}_n)$.*

Proof. According to Theorem 16 ϕ is equivalent to a Boolean combination of sentences and $7^{q-1}/2$-local formulas. Putting it in disjunctive normal form and considering all Boolean combinations of the sentences, we get that ϕ is equivalent to $\bigvee_{i=1}^{N} \theta_i \wedge \psi_i$, for some sentences $\theta_1, \ldots, \theta_N$ and $7^{q-1}/2$-local formulas $\psi_1, \ldots, \psi_N$, with the additional property that in every model exactly one of the sentences θ_i is satisfied. (Formally we require $\vdash \bigvee_i \theta_i$ and $\vdash (\theta_i \to \neg\theta_j)$ for $i \neq j$.) As $(\mathbf{A}_n)_{n \in \mathbb{N}}$ is elementary convergent, there exists $1 \leq a \leq N$ and $n_0 \in \mathbb{N}$ such that $\mathbf{A}_n \models \theta_a$ for every $n \geq n_0$. Let $\widetilde{\phi} = \psi_a$. Then the result follows from $\theta_a \vdash (\phi \leftrightarrow \psi_a)$. □

We can now prove our main result, which directly implies Conjecture 1 and, from which will also follow Conjecture 2.

Theorem 18. *Every quasi-residual FO-convergent sequence has a modeling limit.*

Proof. Let $(A_n)_{n \in \mathbb{N}}$ be an FO-convergent quasi-residual sequence. According to Lemma 12, up to considering a subsequence, there exists an FO-convergent marked quasi-residual sequence $(\mathbf{A}_n^+)_{n \in \mathbb{N}}$ of σ-structures such that $\mathrm{Forget}(\mathbf{A}_n^+) = \mathbf{A}_n$.

Let $\mathbf{L}^+$ be a modeling with properties asserted by Theorem 3, and let $\mathbf{L} = \mathrm{Forget}(\mathbf{L}^+)$. Our aim is to prove that $\mathbf{L}$ is a modeling limit of the sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$.

Let $\phi \in \mathrm{FO}(\sigma)$ be a formula with quantifier rank q and p free variables, and let $\epsilon > 0$ be a positive real.

Let $d = 7^{q-1}/2$ and let m and n_0 be integers such that for every $n \geq n_0$ no ball of radius $8d$ in $\mathrm{Gaifman}(\mathbf{A}_n) \setminus \bigcup_{i=1}^m M_i(\mathbf{A}_n^+)$ contains at least $(\epsilon/p^2)|A_n|$ vertices.

Let $\mathbf{A}_n^* = \mathrm{Encode}_m(\mathbf{A}_n^+)$. Each relation of $\mathbf{A}_n^*$ being defined by a fixed formula from relations of $\mathbf{A}_n^+$, the sequence $(\mathbf{A}_n^*)_{n\in\mathbb{N}}$ is FO-convergent and $\mathbf{L}^* = \mathrm{Encode}_m(\mathbf{L}^+)$ is a modeling FO_1-limit of $(\mathbf{A}_n^*)_{n\in\mathbb{N}}$ satisfying additional properties asserted by Theorem 3.

Let T_m be the elimination theory of $\mathbf{L}^+$ (as defined above). As $\mathbf{L}^+$ is an FO_1-limit (hence an elementary limit) of $(\mathbf{A}_n^+)_{n\in\mathbb{N}}$ there exists $n_1 \geq n_0$ such that for every symbol $R \in \sigma$ with arity $k > 1$ used in ϕ, if $\varsigma_R^{\mathcal{Z}} \in T_m$ then $\mathbf{A}_n^+ \models \varsigma_R^{\mathcal{Z}}$ holds for every $n \geq n_1$. Let $\widehat{\phi}$ be the elimination formula of ϕ with respect to T_m. Note that $\widehat{\phi}$ has also quantifier rank at most q. According to Lemma 15, for every $n \geq n_1$ it holds $\widehat{\phi}(\mathbf{A}_n^*) = \phi(\mathbf{A}_n^+)$. Thus, as $\phi(\mathbf{A}_n^+) = \phi(\mathbf{A}_n)$ (as ϕ only uses symbols in σ) it holds

$$\forall n \geq n_1 \qquad \langle \widehat{\phi}, \mathbf{A}_n^* \rangle = \langle \phi, \mathbf{A}_n \rangle. \tag{9}$$

As $\mathbf{L}^*$ satisfies T_m we get

$$\langle \widehat{\phi}, \mathbf{L}^* \rangle = \langle \phi, \mathbf{L} \rangle. \tag{10}$$

Note that by our choice of m the sequence $(\mathbf{A}_n^*)$ is $(8d, \epsilon/p^2)$-residual hence by Lemma 14 the modeling $\mathbf{L}^*$ is $(2d, \epsilon/p^2)$-residual.

According to Lemma 17 there exists a d-local formula $\widetilde{\phi}$ and an integer $n_2 \geq n_1$ such that for every $n \geq n_2$ it holds $\widehat{\phi}(\mathbf{A}_n^*) = \widetilde{\phi}(\mathbf{A}_n^*)$ hence

$$\forall n \geq n_2 \qquad \langle \widetilde{\phi}, \mathbf{A}_n^* \rangle = \langle \phi, \mathbf{A}_n \rangle. \tag{11}$$

As $\mathbf{L}^*$ is elementary limit of $(\mathbf{A}_n^*)_{n\in\mathbb{N}}$ it similarly holds

$$\langle \widetilde{\phi}, \mathbf{L}^* \rangle = \langle \phi, \mathbf{L} \rangle. \tag{12}$$

According to Lemma 10 (as $\widetilde{\phi}$ is d-local, $(\mathbf{A}_n^*)$ is $(8d, \epsilon/p^2)$-residual and $\mathbf{L}^*$ is $(2d, \epsilon/p^2)$-residual) it holds

$$|\langle \widetilde{\phi}, \mathbf{L}^* \rangle - \lim_{n\to\infty} \langle \widetilde{\phi}, \mathbf{A}_n^* \rangle| < \epsilon.$$

Hence by (11) and (12) it holds

$$|\langle \phi, \mathbf{L} \rangle - \lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle| < \epsilon. \tag{13}$$

As (13) holds for every $\epsilon > 0$ we have

$$\langle \phi, \mathbf{L} \rangle = \lim_{n\to\infty} \langle \phi, \mathbf{A}_n \rangle.$$

As this holds for every formula $\phi \in \mathrm{FO}(\sigma)$, we conclude that $\mathbf{L}$ is a modeling limit of $(\mathbf{A}_n)_{n\in\mathbb{N}}$. □

From Theorems 7 it follows that any FO-convergent sequence of graphs from a nowhere dense class is quasi-residual thus from Theorem 18 directly follows a proof of Conjecture 2.

Corollary 3. *Let $\mathcal{C}$ be a monotone class of graphs. Then $\mathcal{C}$ has modeling limits if and only if $\mathcal{C}$ is nowhere dense.*

6. Further Comments

6.1. **Approximation.** Let A and B be measurable subsets of the domain L of the modeling limit of an FO-convergent sequence $(\mathbf{A}_n)_{n\in\mathbb{N}}$ of finite structures. Assume that every element in A has at least b neighbours in B and every element in B has at most a neighbours in A.

The *strong finitary mass transport principle* asserts that in such a case it should hold

$$b\,\nu_{\mathbf{L}}(A) \le a\,\nu_{\mathbf{L}}(B). \tag{14}$$

It is easily checked that if both A and B are first-order definable (without parameters) then (14) holds: let $A = \phi(\mathbf{L})$ and $B = \psi(\mathbf{L})$. Define

$$\phi'(x) := \phi(x) \wedge (\exists y_1 \dots y_b) \bigwedge_{i=1}^{b} \Big((y_i \sim x) \wedge \psi(y_i) \wedge \bigwedge_{i<j\le b} (y_i \neq y_j)\Big)$$

$$\psi'(x) := \psi(x) \wedge \neg(\exists y_1 \dots y_{a+1}) \bigwedge_{i=1}^{a+1} \Big((y_i \sim x) \wedge \phi(y_i) \wedge \bigwedge_{i<j\le a+1} (y_i \neq y_j)\Big)$$

Then $\nu_{\mathbf{L}}(A) = \nu_{\mathbf{L}}(\phi'(\mathbf{L}))$ and $\nu_{\mathbf{L}}(B) = \nu_{\mathbf{L}}(\psi'(\mathbf{L}))$. As $b\,\langle\phi', \mathbf{A}_n\rangle \le a\,\langle\psi', \mathbf{A}_n\rangle$ holds for every integer n (as $\mathbf{A}_n$ is finite), by continuity we deduce $b\,\nu_{\mathbf{L}}(A) \le a\,\nu_{\mathbf{L}}(B)$.

However, it is not clear whether an FO-convergent sequence of graphs from a nowhere dense class has a modeling limit that satisfies the strong finitary mass transport principle. This can be formulated as

Conjecture 3. One can require a version of the strong mass transport principle.

6.2. **Characterization.** In this context, it is natural to propose the following generalization of Aldous-Lyons conjecture.

Conjecture 4. Let $\mathbf{L}$ be a modeling such that:

- the theory of $\mathbf{L}$ has the finite model property.
- every interpretation of $\mathbf{L}$ satisfies the finitary mass transport principle. Precisely, for every first-order formulas α, β, γ such that

$$\alpha(x) \vdash (\exists y_1 \dots y_b) \bigwedge_{i=1}^{b} \Big(\gamma(y_i, x) \wedge \beta(y_i) \wedge \bigwedge_{i<j\le b} (y_i \neq y_j)\Big)$$

$$\beta(x) \vdash \neg(\exists y_1 \dots y_{a+1}) \bigwedge_{i=1}^{a+1} \Big(\gamma(x, y_i) \wedge \alpha(y_i) \wedge \bigwedge_{i<j\le a+1} (y_i \neq y_j)\Big)$$

  it holds

$$b\,\langle\alpha, \mathbf{L}\rangle \le a\,\langle\beta, \mathbf{L}\rangle.$$

- for every integer d there is an integer N such that $\mathbf{L}$ does not contain the d-th subdivision of K_N.

Then $\mathbf{L}$ is the FO-limit of a sequence of finite graphs.

Note that there may be weaker versions of the finitary mass transport principle non-trivially equivalent for it. See for instance what happens with mappings [32].

Note that the last condition implies that there exists no integer d such that $\mathbf{L}$ includes the d-subdivision of $K_{\aleph_0,2^{\aleph_0}}$, thus $\mathbf{L}$ has a countable skeleton, that is there are $s_1,\dots,s_n,\dots\in L$ such that for every integer d and every $\epsilon>0$ there is N with the property

$$\sup_{v\in L-\{s_1,\dots,s_N\}}\nu_{\mathbf{L}}(B_d(\mathbf{L}-\{s_1,\dots,s_N\},v))\leq\epsilon.$$

6.3. $\mathcal{L}(Q_m)$-Theory of Modelings.

Conjecture 5. For a modeling $\mathbf{A}$, the knowledge of all $\langle\phi,\mathbf{A}\rangle$ (for first-order formulas ϕ) is sufficient to deduce the complete $\mathcal{L}(Q_m)$-theory of $\mathbf{A}$.

As a support for Conjecture 5 consider the following $\mathcal{L}(Q_m)$ sentences (where ϕ is a first-order formula):

$$\begin{aligned}\Phi:&\qquad(\exists y)\,(Q_m x)\,\phi(x,y)\\ \Psi:&\qquad(\forall y)\,(Q_m x)\,\phi(x,y)\end{aligned}$$

Then it is easily checked that

$$\begin{aligned}\mathbf{M}\models\Phi &\iff \langle(\exists y)\,\phi(x_1,y)\wedge\phi(x_2,y),\mathbf{M}\rangle>0\\ \mathbf{M}\models\Psi &\iff \lim_{k\to\infty}\langle(\exists y)\,\neg\phi(x_1,y)\vee\dots\vee\neg\phi(x_k,y),\mathbf{M}\rangle^{1/k}=0\end{aligned}$$

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