Characterizing subclasses of cover-incomparability graphs by forbidden subposets

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Abstract. In this paper we demonstrate that several theorems from [1] and [3] do not hold as they are stated. These are the theorems regarding forbidden subposet characterizations of certain classes of cover- incomparability graphs. In this paper we correct a mistake in the main theorem of [1], reformulate the corresponding statements and present corrected proofs. We further characterize posets whose cover-incomparability graphs are interval graphs and unit interval graphs.

1 Introduction

In this paper we deal with posets and graphs associated to them. There are several ways how to associate a graph G to a given poset P. The vertex set $V(G)$ is usually the set of points of P. Depending on the edge-set $E(G)$, we may obtain among others the *comparability graph* of P (x and y are adjacent iff $x < y$ or $y < x$), the *incomparability graph* of P (x and y are adjacent iff x and y are incomparable), the *cover graph* of P (x and y are adjacent iff x covers y or vice versa) or the *cover-incomparability graph* of P (x and y are adjacent iff x covers y, or y covers x, or x and y are incomparable). The incomparability graph of P is of course just the complement of its comparability graph, while the cover-incomparability graph of P is the union of the cover graph and the incomparability graph of G.

Cover graphs, comparability graphs and incomparability graphs are standard ways how to associate a graph to a given poset, while the notion of cover-incomparability graph is new. It was introduced in [1]. This notion was motivated by the theory of transit functions on posets. It turns out that the underlying graph G_P of the standard transit function T_P on the poset P is exactly the cover-incomparability graph of P (see [1] for details).

Cover-incomparability graphs have been sofar approached in two different ways. One possibility is to try to characterize graphs that are cover-incomparability graphs. In [6] it was proved that the recognition problem for cover-incomparability graphs is in general NP-complete. On the other hand there are classes of graphs (such as trees, Ptolemaic graphs, distance-hereditary graphs, block graphs, split graphs or k-trees) for which the recognition problem can be solved in linear time (see [2,3,7,8] for details and proofs).

Another approach is to study posets whose cover-incomparability graphs have certain property. Posets whose cover-incomparability graphs are chordal, Ptolemaic, distance-hereditary, claw-free or cographs were characterized in [1] and [4]. Unfortunately, there is a mistake that originated in [1] and continued in [4] and several statements from these papers do not hold as they are stated. In this paper we correct the mistake and reformulate the corresponding statements so that they hold.

It is known (see e.g. [10]) that a graph G is interval if and only if G is chordal and the complement $\overline{G}$ admits a transitive orientation. As for any cover-incomparability graph G its complement $\overline{G}$ must admit a transitive orientation [6], it follows that a cover-incomparability graph is interval if and only if it is chordal. In Section 5 we present another proof of this statement. Instead of using the characterization of interval graphs from [10] (G is interval if and only if G is chordal and the complement $\overline{G}$ admits a transitive orientation.), we start from the characterization of interval graphs by forbidden induced subgraphs [5] and obtain a characterization of posets whose cover-incomparability graphs are interval. As this characterization is the same as for chordal graph, we immediately obtain that a cover-incomparability graph is interval if and only if it is chordal.

Our paper is organized as follows. In Section 2 we give an overview of terminology and basic properties of cover-incomparability graphs. In Section 3 we present counterexamples to Theorem 4.1 from [3], Lemma 4.4 and 4.5 from [1] and to Proposition 5.1 from [1]. In Section 4 we show that the mistake originated in Theorem 2.4 [1]. We reformulate this statement and give a corrected proof of it. In addition, we reformulate all the above mentioned statements so that they hold. In Section 5 we characterize posets whose cover- incomparability graphs interval graphs and unit interval graphs.

2 Terminology and basic properties

Let $P = (V, \leq)$ be a poset. We will use the following notation. For $u, v \in V$ we write:

- $u < v$ if $u \leq v$ and $u \neq v$.
- $u \lhd v$ if $u < v$ and there is no $z \in V$ such that $u < z < v$. We say that v *covers* u.
- $u \lhd\lhd v$ if $u < v$ and $\neg(u \lhd v)$.
- $u \parallel v$ if u and v are incomparable.

Definition 1. *For a given poset $P = (V, \leq)$, let $G_P = (V, E)$ be a graph with $E = \{\{u, v\} \mid u \lhd v$ or $v \lhd u$ or $u \parallel v\}$. Then we say that G_P is the* cover-incomparability graph *of P (or the C-I graph of P for short).*

Note that for any $u, v \in V(G_P)$, $u \neq v$ we have $\{u, v\} \notin E(G_P) \Leftrightarrow u \lhd \lhd v$ or $v \lhd \lhd u$.

As this is crucial for the rest of our paper let us define properly the following three concepts.

Definition 2. *Let $P = (V_P, \leq_P)$ be a poset.*

- *We say that $Q = (V_Q, \leq_Q)$ is a* subposet *of $P = (V_P, \leq_P)$ if*
  1. *$V_Q \subseteq V_P$ and*
  2. *for any $u, v \in V_Q$ we have $u \leq_Q v \Leftrightarrow u \leq_P v$.*
- *We say that $R = (V_R, \leq_R)$ is an* isometric subposet *of $P = (V_P, \leq_P)$ if*
  1. *$V_R \subseteq V_P$ and*
  2. *for any $u, v \in V_R$ we have $u \leq_R v \Leftrightarrow u \leq_P v$ and*
  3. *for any $u, v \in V_R$ such that $u \leq_R v$ a chain of a shortest length between u and v in P is also in R.*
- *We say that $S = (V_S, \leq_S)$ is a* $\lhd$-preserving subposet *of $P = (V_P, \leq_P)$ if*
  1. *$V_S \subseteq V_P$ and*
  2. *for any $u, v \in V_S$ we have $u \leq_S v \Leftrightarrow u \leq_P v$ and*
  3. *for any $u, v \in V_S$ we have $u \lhd_S v \Leftrightarrow u \lhd_P v$.*

Note that an isometric subposet is always $\lhd$-preserving but there are $\lhd$-preserving subposets that are not isometric. For example, the poset P' depicted in Fig. 1 is a nonisometric $\lhd$-preserving subposet of P in Fig. 1.

Fig. 1: A nonisometric $\lhd$-preserving subposet.

Let us also mention a few easy observations about C-I graphs. They follow immediately from the definition.

Lemma 1. *Let $P = (V, \leq)$ be a poset and $G_P = (V, E)$ its C-I graph. Then the following holds.*

(i) G_P is connected.
(ii) If $U \subseteq V$ is an antichain in P then U induces a complete subgraph in G_P.

(iii) *If $I \subseteq V$ is an independent set in G_P then all points of I lie on a common chain in P.*
(iv) *There are at most 2 vertices of degree 1 in G_P.*
(v) *If $P^* = (V, \leq^*)$ is the dual poset to P (i.e. $u \leq v$ in $P \Leftrightarrow v \leq^* u$ in P^*), then $G(P^*) = G_P$.*
(vi) *If the vertices x, y, z form a triangle in G_P then at least two of them are incomparable.*
(vii) *Let x, y, z be vertices of G_P such that $xy \in E$, $xz \notin E$, $yz \notin E$. Then ($x \lhd\lhd z$ and $y \lhd\lhd z$) or ($z \lhd\lhd x$ and $z \lhd\lhd y$).*

3 Counterexamples

In this section we present counterexamples to several statements from [1] and [3]. Let us start with the easiest case, with Proposition 5.1 from [1].

3.1 A counterexample to Proposition 5.1 from [1]

First we cite the statement of this proposition in the original text:

Proposition (Proposition 5.1 [1]). Let P be a poset. Then G_P contains an induced claw if and only if P contains one of S_1, S_2 or S_3 as an isometric subposet, see Fig. 2.

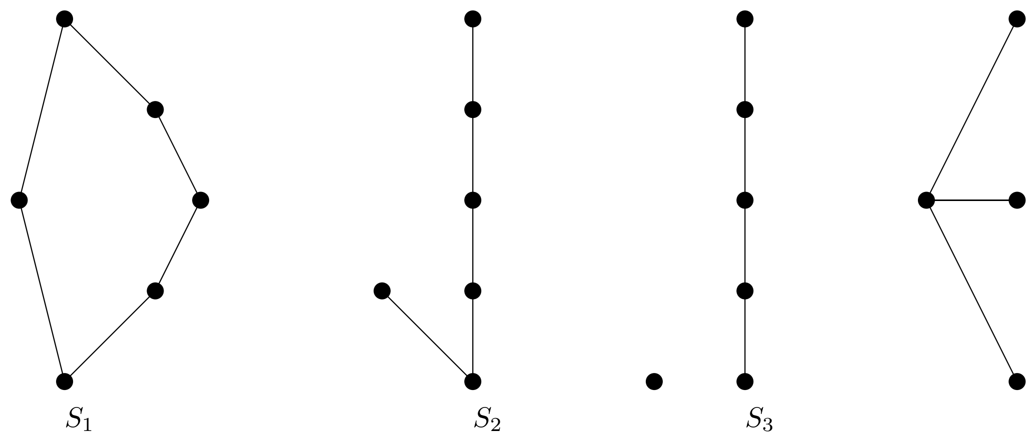

Fig. 2: Subposets S_1, S_2 and S_3 and the claw.

This statement does not hold. Let P be the poset depicted in Fig. 3. Clearly, neither S_1 nor S_2 are subposets of P. S_3 is a subposet of P but it is *not an isometric subposet* of P. This is because there is a chain of length 2 between u and v in P while is no chain of length 2 between u and v in S_3. Thus P does not contain any of S_1, S_2 and S_3 as an isometric subposet. But G_P contains an induced claw on vertices v, v_1, v_3, v_5, a contradiction.

Counterexamples to other statements can be derived in a similar way:

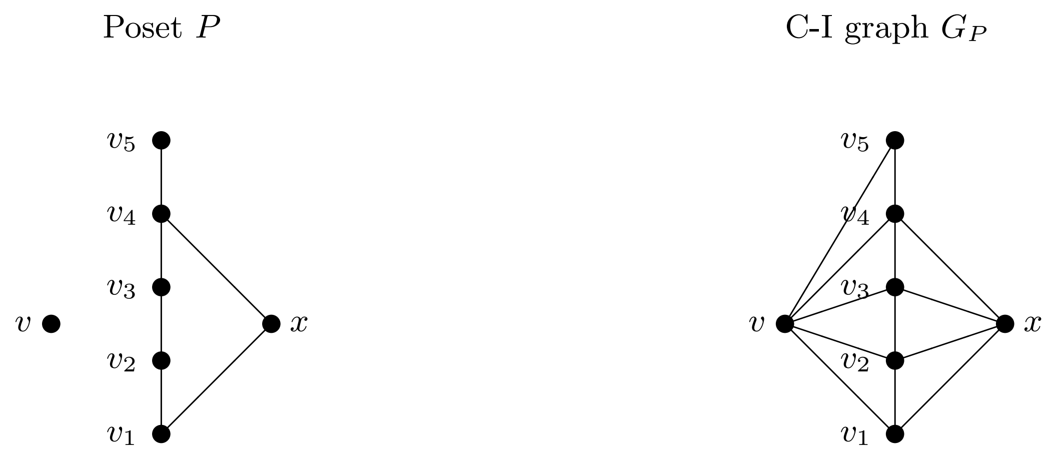

Fig. 3: A counterexample to Proposition 5.1.

3.2 A counterexample to Lemma 4.4 from [1]

Proposition (Proposition 4.4 in [1]). Let P be a poset. Then G_P contains an induced house if and only if P contains one of R_1, R_2, R_3, R_4 or R_5 as an isometric subposet, see Figure 4.

Let P be the poset depicted in Fig. 5. It is easy to see that it is a counterexample to Lemma 4.4 [1]. Indeed, P does not contain any of the posets R_1, R_2, R_3, R_4 or R_5 as an **isometric subposet**. But G_P contains an induced house on vertices v_1, v_2, v_4, v_5, v_7, a contradiction.

3.3 A counterexample to Lemma 4.5 from [1]

Proposition (Proposition 4.5 in [1]). Let P be a poset. Then G_P contains an induced domino if and only if P contains one of D_1, D_2, D_3, D_4, D_5, D_6 or D_7 as an isometric subposet, see Fig. 6.

Let P be the poset depicted in Figure 7. Clearly that it is a counterexample to Lemma 4.5. [1]. Indeed, P does not contain any of the posets D_1, D_2, D_3, D_4, D_5, D_6 or D_7 as an *isometric subposet.* But G_P contains an induced domino on vertices v_1, v_2, v_4, v_5, v_7, a contradiction.

3.4 A counterexample to Theorem 4.1 from [3]

Theorem (Theorem 4.1 [3]). Let P be a poset. Then G_P is a cograph if and only if P contains neither any of Q_1, Q_2, ..., Q_7 nor duals of Q_2 or Q_5 as an isometric subposet, see Fig. 8.

Let P be the poset depicted in Fig. 7. It is easy to see that it is a counterexample to Theorem 4.1 [3]. Indeed, P contains neither any of the posets Q_1, Q_2, ..., Q_7 nor the duals of Q_2 or Q_5 as an *isometric subposet.* But G_P contains an induced path on four vertices v_1, v_2, v_3, v_4. Thus, G_P is not a cograph, a contradiction.

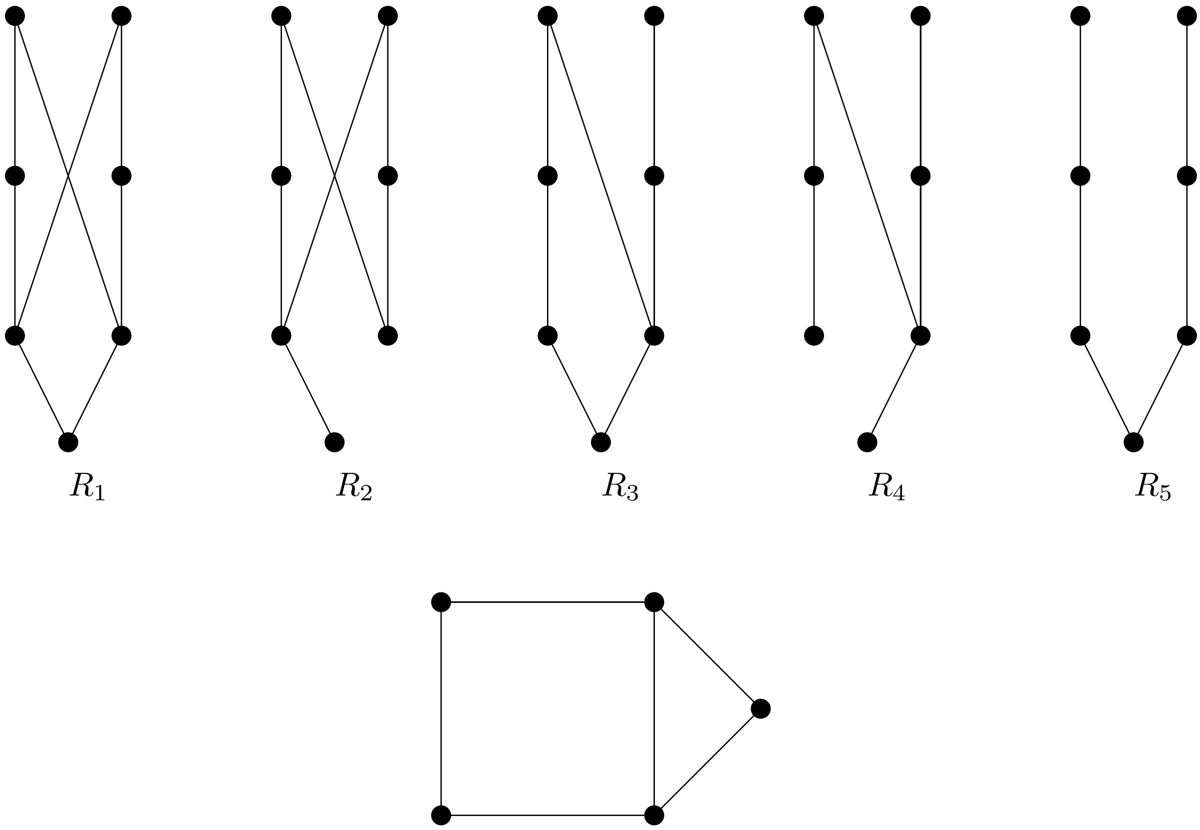

Fig. 4: Subposets R_i, $i = 1, \ldots 5$ and the house.

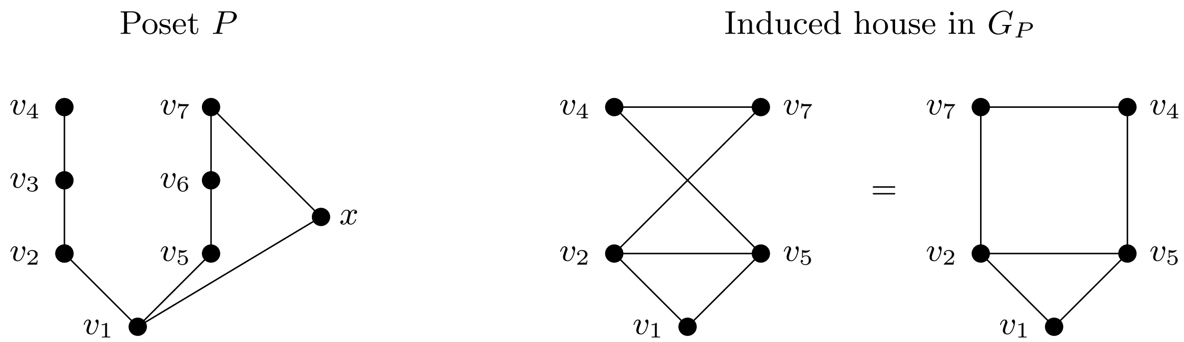

Fig. 5: A counterexample to Lemma 4.4.

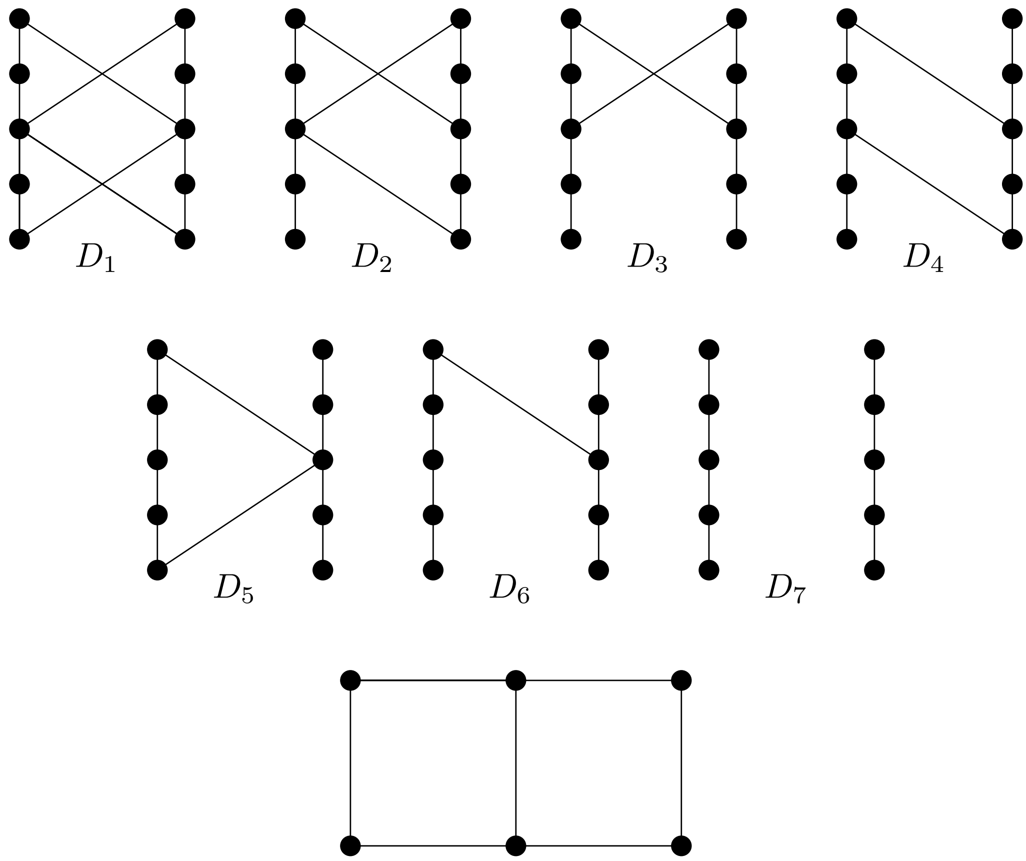

Fig. 6: Subposets D_i, $i = 1, \dots 7$, and the domino graph.

Fig. 7: A counterexample to Lemma 4.5.

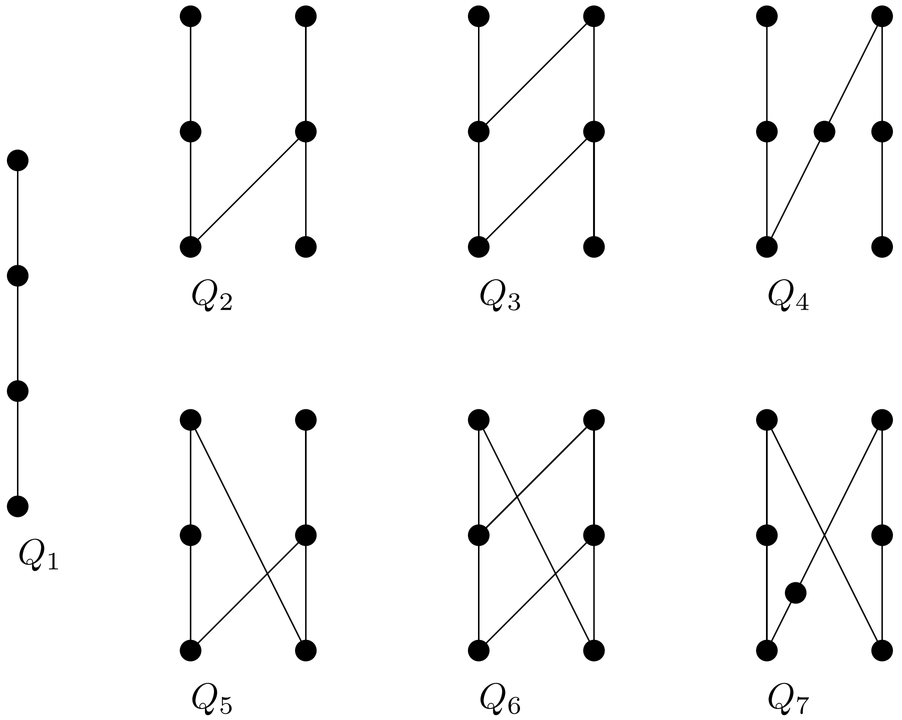

Fig. 8: Subposets Q_i, $i = 1, \dots 7$.

Fig. 9: A counterexample to Theorem 4.1

4 Restatements and proofs

The mistake originated in Theorem 2.4 [1].

Theorem (Theorem 2.4 [1]). Let $\mathcal{G}$ be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathcal{P} = \{P \mid P \text{ is a poset with } G_P \in \mathcal{G}\}$. Then $\mathcal{P}$ has a forbidden isometric characterization.

If we go carefully through the proof of this theorem in [1] we notice that it is not proved that the poset P contains one of the constructed posets $\{P_i\}_{i\in I}$ as an *isometric* subposet. The condition of isometry is too strong and it has to be replaced by the weaker concept of $\lhd$-preserving subposet. See Section 2 for the definition.

Theorem 1 (corrected version of Theorem 2.4 [1]). *Let $\mathcal{G}$ be a class of graphs with a characterization by forbidden induced subgraphs. Let $\mathcal{P} = \{P \mid P \text{ is a poset with } G_P \in \mathcal{G}\}$. Then $\mathcal{P}$ has a characterization by forbidden $\lhd$-preserving subposets.*

For the proof of this theorem we need a slightly stronger version of Lemma 2.3 [1].

Lemma 2 (corrected version of Lemma 2.3 [1]). *Let Q be a $\lhd$-preserving subposet of a poset P. Then G_Q is isomorphic to a subgraph of G_P induced by the points of Q.*

Proof. Let H be the subgraph of G_P induced by the points of Q. Let u and v be arbitrary points in Q. We show that

$$\{u, v\} \in E(H) \Leftrightarrow \{u, v\} \in E(G_Q).$$

First suppose that $\{u, v\} \in E(H)$. This happens if and only if either $u \lhd_P v$, or $v \lhd_P u$, or $u \parallel_P v$. As Q is a $\lhd$-preserving subposet of P we have

$$\begin{aligned} u \lhd_P v &\Rightarrow u \lhd_Q v \Rightarrow \{u, v\} \in E(G_Q), \\ v \lhd_P u &\Rightarrow v \lhd_Q u \Rightarrow \{u, v\} \in E(G_Q), \\ u \parallel_P v &\Rightarrow u \parallel_Q v \Rightarrow \{u, v\} \in E(G_Q). \end{aligned}$$

Thus if $\{u, v\} \in E(H)$ then also $\{u, v\} \in E(G_Q)$.

Now suppose that $\{u, v\} \notin E(H)$. Then $u \lhd \lhd_P v$ or $v \lhd \lhd_P u$. As Q is a $\lhd$-preserving subposet of P it follows that $u \lhd \lhd_Q v$ or $v \lhd \lhd_Q u$, and thus $\{u, v\} \notin E(G_Q)$.

We conclude that H and G_Q are isomorphic graphs as stated. $\square$

Now we are ready to prove Theorem 1.

Proof ((of Theorem 1)). Let G_{forb} be one of the forbidden induced subgraphs for the class $\mathcal{G}$. Let $P \in \mathcal{P}$ be any poset in the class $\mathcal{P}$. By the definition of $\mathcal{P}$, G_P does not contain G_{forb} as an induced subgraph. By Lemma 2, P does not contain any $\lhd$-preserving subposet Q such that G_Q is isomorphic to G_{forb}. Hence any subposet Q s.t. G_Q is isomorphic to G_{forb} is forbidden for $\mathcal{P}$. Repeating this for all the forbidden induced subgraphs for $\mathcal{G}$ we find a list of forbidden $\lhd$-preserving subposets $\{Q_i\}_{i\in I}$.

We will show that the class $\mathcal{P}$ is characterized by forbidden $\lhd$-preserving subposets $\{Q_i\}_{i\in I}$.

First, let $P \in \mathcal{P}$. Then P clearly contains no Q_i as a $\lhd$-preserving subposet. Otherwise (by Lemma 2) the graph G_P would contain a forbidden induced subgraph for $\mathcal{G}$.

Conversely, suppose that P contains no Q_i as a $\lhd$-preserving subposet. Then (by the construction of $\{Q_i\}_{i\in I}$) G_P contains no forbidden subgraph for $\mathcal{G}$. Thus $G_P \in \mathcal{G}$, and hence $P \in \mathcal{P}$. □

The previous theorem can be applied for various graph classes that admit a characterization by forbidden induced subgraphs, such as chordal graphs, claw-free graphs, distance-hereditary graphs, Ptolemaic graphs etc.

Theorem 2 (corrected Proposition 5.1 [1]). *Let P be a poset. Then G_P contains an induced claw if and only if P contains one of S_1, S_2, S_3 or S_2^* (the dual of S_2) as a $\lhd$-preserving subposet, see Fig. 2.*

Proof. If P contains one of the posets S_1, S_2, S_3 or S_2^* as a $\lhd$-preserving subposet then clearly G_P contains an induced claw.

Conversely, suppose that G_P contains an induced claw. We want to find S_1, S_2, S_3 or S_2^* as a $\lhd$-preserving subposet of P. Let us denote by x the middle vertex and by u, v, w the other vertices of the claw. By Lemma 1(iii), as u, v, w form an independent set in G_P they lie on a common chain in P. Without loss of generality we may suppose that $u \lhd\lhd v \lhd\lhd w$.

Note that $x \lhd v$ is not possible, otherwise $x \lhd\lhd w$ and hence $\{x, w\} \notin E(G_P)$, a contradiction. Similarly, it is not possible that $x \lhd u$, $v \lhd x$ or $w \lhd x$. Thus there are only five cases to distinguish:

- *Case 1* $x \parallel u$, $x \parallel v$, $x \parallel w$. Then P obviously contains S_3 as a $\lhd$-preserving subposet.
- *Case 2* $u \lhd x$, $x \parallel v$, $x \parallel w$. Then P obviously contains S_2 as a $\lhd$-preserving subposet.
- *Case 3* $x \lhd w$, $x \parallel v$, $x \parallel u$. Then P obviously contains S_2^* as a $\lhd$-preserving subposet.
- *Case 4* $u \lhd x$, $x \lhd w$, $x \parallel v$ and the length of the shortest chain in P between u and w is equal to 4. Then P obviously contains S_3 as a $\lhd$-preserving subposet.
- *Case 5* $u \lhd x$, $x \lhd w$, $x \parallel v$ and the length of the shortest chain in P between u and w is greater than 4. Then P obviously contains S_2 as a $\lhd$-preserving subposet.

□

Now let us restate the corresponding statements from [1] and [3]. We skip their proofs as they are the same as the ones presented in [1] and [3], the only mistake was claiming that the forbidden subposets must be isometric subposets of P.

Theorem 3 (corrected Proposition 4.4 [1]). *Let P be a poset. Then G_P contains an induced house if and only if P contains one of R_1, R_2, R_3, R_4, R_5 or its duals as a $\triangleleft$-preserving subposet, see Fig. 4.*

Theorem 4 (corrected Proposition 4.5 [1]). *Let P be a poset. Then G_P contains an induced domino if and only if P contains one of D_1, D_2, D_3, D_4, D_5, D_6, D_7 or its duals as a $\triangleleft$-preserving subposet, see Fig. 6.*

Theorem 5 (corrected Theorem 4.1 [3]). *Let P be a poset. Then G_P is a cograph if and only if P contains neither any of Q_1, Q_2, $\ldots$, Q_7 nor duals of Q_2 or Q_5 as a $\triangleleft$-preserving subposet, see Fig. 8.*

Theorem 6 (reformulated Theorem 3.1 [3]). *Let P be a poset. Then G_P is a chordal if and only if P contains neither any of P_1, P_2, P_3 nor P_2^* (the dual of P_2) as a $\triangleleft$-preserving subposet, see Fig. 10.*

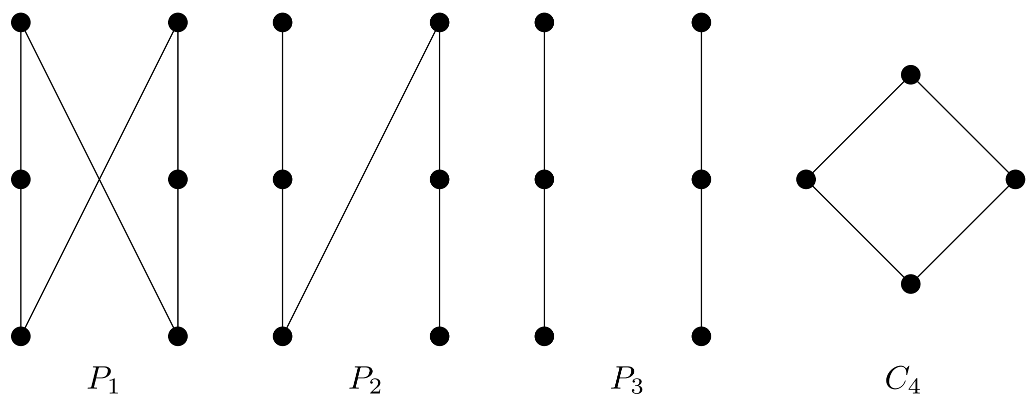

Fig. 10: Subposets P_1, P_2, P_3 and C_4

Let us remark that for P_1, P_2, P_3 and P_2^* the notion of isometric subposet and $\triangleleft$-preserving subposet coincide. More precisely, a poset P contains P_1, P_2, P_3 or P_2^* as an isometric subposet if and only if P contains P_1, P_2, P_3 or P_2^* as a $\triangleleft$-preserving subposet. This is because the length of the longest chain in P_1, P_2, P_3 and P_2^* is only two. Hence, Theorem 3.1. [3] holds as it was stated in [3].

5 Interval and unit interval C-I graphs

In this section we characterize posets whose cover incomparability graphs are interval or unit interval graphs. We start from the characterization of interval graphs by forbidden induced subgraphs [5].

Interval graphs are the intersection graphs of a set of intervals on the real line. That is, the graph has one vertex for each interval in the set, and an edge between every pair of vertices corresponding to intervals that intersect. *Unit interval graphs* are interval graphs that can be represented by intervals of the

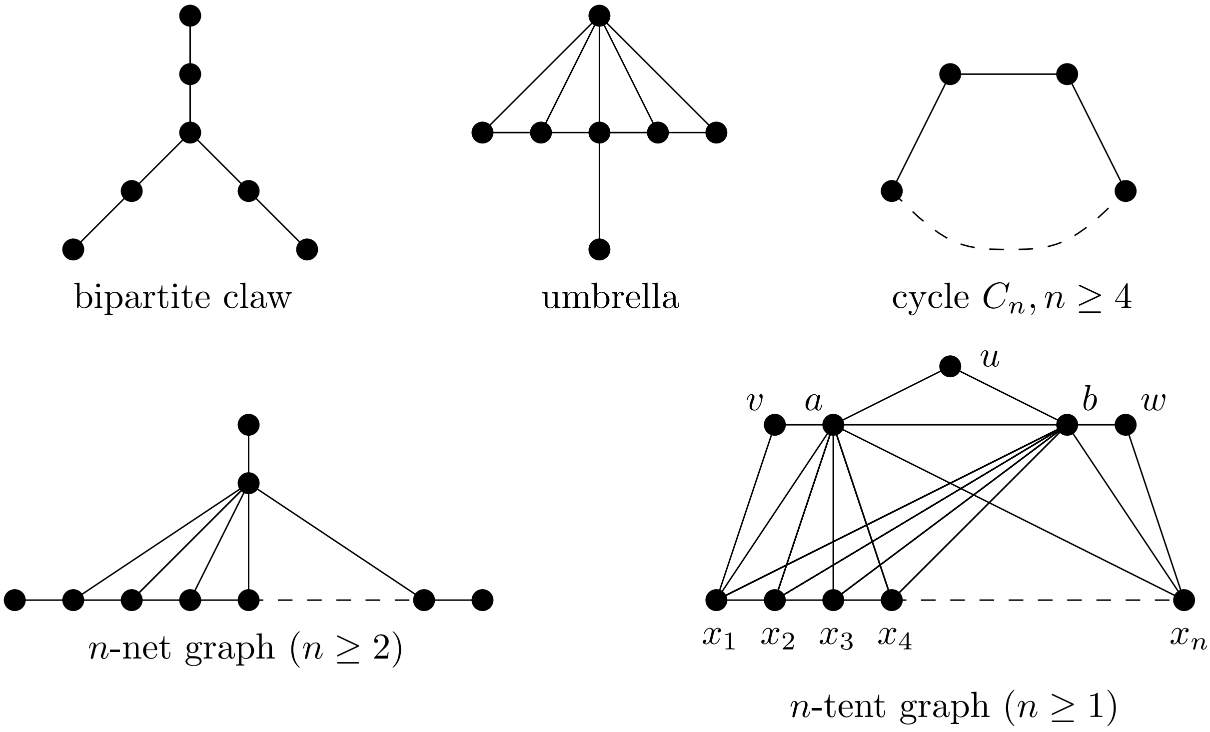

Fig. 11: Forbidden induced subgraphs for the class of interval graphs.

same length. Interval graphs (resp. unit interval graphs) can be characterized by a set of forbidden induced subgraphs.

Theorem 7 ([5]). *A graph G is an interval graph if and only if G does not contain any of the following graphs (depicted in Fig. 11) as an induced subgraph: the bipartite claw, n-net for $n \geq 2$, umbrella, n-tent for $n \geq 3$, and cycle C_n for $n \geq 4$.*

It is well known since the seminal paper of Roberts [9] that an interval graph G is unit interval if and only if it does not contain induced $K_{1,3}$. Thus, by excluding graphs that contain induced $K_{1,3}$ from the graphs depicted in Fig. 11, we immediately obtain the set of forbidden induced subgraphs for unit interval graphs:

Corollary 1. *A graph is a unit interval graph if and only if it contains no induced $K_{1,3}$, 2-net, 3-tent, or C_n, for any $n \geq 4$.*

For an undirected graph $G = (V, E)$ an orientation G is the oriented graph $D = (V, A)$ that arises from G by replacing each edge uv by one of the arcs $\overrightarrow{uv}$ or $\overrightarrow{vu}$. The orientation D is said to be *transitive* if for any two consecutive arcs $\overrightarrow{uv} \in A(D)$, $\overrightarrow{vw} \in A(D)$ also $\overrightarrow{uw} \in A(D)$.

Now, let us turn our attention back to cover-incomparability graphs. If H is an induced subgraph of a C-I graph G then the complement $\overline{H}$ admits a transitive orientation. Indeed, if $uv \notin E(H)$ (and thus also $uv \notin E(G_P)$) then $u \lhd\lhd v$ or $v \lhd\lhd u$ in P. We define an orientation D on the non-edges of H (i.e. edges of $\overline{H}$) by

$$\overrightarrow{uv} \in A(D) \Leftrightarrow u \lhd\lhd v.$$

As the relation $\lhd\lhd$ is transitive, D is a transitive orientation of $\overline{H}$. In fact, H is an induced subgraph of a C-I graph G if and only if the complement $\overline{H}$ admits a transitive orientation (see [6] for the proof of the other implication).

Now we will prove that the complements of the most of the above mentioned forbidden induced subgraphs do not admit a transitive orientation, and hence cannot occur as induced subgraphs of any C-I graph. We will repeatedly use the following easy observation.

Lemma 3. *Let G be a graph and let D be a transitive orientation of the $\overline{G}$. Let u, v, w be vertices such that $\{u, v\} \in E(G)$, $\{u, w\} \notin E(G)$ and $\{v, w\} \notin E(G)$. If $(u, w) \in D$ then also $(v, w) \in D$. If $(w, u) \in D$ then also $(w, v) \in D$.*

Lemma 4. *The complement of the bipartite claw does not admit a transitive orientation.*

Proof. Let us denote the vertices of the bipartite claw by x_1, x_2, y_1, y_2, z_1, z_2 and c as it is depicted in Fig. 11. Suppose for contradiction that the complement graph admits a transitive orientation D. Without loss of generality we may assume that $(x_2, y_2) \in D$. By successive application of Lemma 3 we get that $(x_2, y_1) \in D$, $(x_2, c) \in D$, $(x_2, z_1) \in D$, $(x_2, z_2) \in D$. Now again by successive application of Lemma 3 we get that $(x_1, z_2) \in D$, $(c, z_2) \in D$, $(y_1, z_2) \in D$, $(y_2, z_2) \in D$. And similarly that $(x_1, y_2) \in D$, $(c, y_2) \in D$, $(z_1, y_2) \in D$, $(z_2, y_2) \in D$. Thus we have that both and $(y_2, z_2) \in D$ and $(z_2, y_2) \in D$, a contradiction. □

Lemma 5. *The complement of the n-net does not admit a transitive orientation for any $n \geq 2$.*

Proof. Let us denote the vertices of the n-net, $n \geq 2$, by u, v, w $x_1, \ldots x_n$ and c as it is depicted in Fig. 11. Suppose for contradiction that the complement graph admits a transitive orientation D. Without loss of generality we may assume that $(u, v) \in D$. By successive application of Lemma 3 we get that $(x_1, v), \ldots, (x_{n-1}, v) \in D$. Furthermore $(c, v) \in D$ and $(w, v) \in D$. Similarly, we get that $(u, x_n), \ldots, (u, x_2) \in D$. Also $(u, c) \in D$ and $(u, w) \in D$. That implies $(x_1, w), \ldots, (x_n, w) \in D$ and $(v, w) \in D$, a contradiction. □

Lemma 6. *The complement of the umbrella does not admit a transitive orientation.*

Proof. Let us denote the vertices of the umbrella by u, $x_1, \ldots x_5$ and v as it is depicted in Fig. 11. Suppose for contradiction that the complement graph admits a transitive orientation D. Without loss of generality we may assume that $(u, v) \in D$. By successive application of Lemma 3 we get that $(u, x_1), (u, x_2), (u, x_4), (u, x_5) \in D$. It follows that $(x_3, x_1) \in D$ and $(x_3, x_5) \in D$. Furthermore $(x_3, x_1) \in D$ implies that $(x_4, x_1) \in D$ and $(x_5, x_1) \in D$ while $(x_3, x_5) \in D$ implies that $(x_2, x_5) \in D$ and $(x_1, x_5) \in D$, a contradiction. □

Lemma 7. *The complement of the n-tent does not admit a transitive orientation for any $n \geq 3$.*

Proof. Let us denote the vertices of the n-tent, $n \geq 3$, by $x_1, \dots x_n$, u, v, w, a and b as it is depicted in Fig. 11. Suppose for contradiction that the complement graph admits a transitive orientation D. Without loss of generality we may assume that $(u, v) \in D$. By successive application of Lemma 3 we get that $(u, x_1) \in D$, $(u, x_2) \in D$, $\dots (u, x_n) \in D$, $(u, w) \in D$. Now by applying Lemma 3 to the triple of vertices u, v, b, we get that $(b, v) \in D$. Similarly, by applying Lemma 3 to the triple u, w, a, we get that $(a, w) \in D$. Now we apply Lemma 3 to the triple a, v, w and get $(v, w) \in D$. Similarly, by applying Lemma 3 to the triple b, v, w, we get $(w, v) \in D$, a contradiction. □

For the proof of the following lemma see [1] or [6].

Lemma 8. *Let G be a graph that contains C_n, $n \geq 5$ as an induced subgraph. Then G cannot be the C-I graph of any poset.*

Using Lemmas 4 up to 8 we get that from all the posets depicted in Fig. 11 only C_4 can appear as induced subgraph of a C-I graph. Combining this fact with Theorem 6 we get the following statement.

Theorem 8. *Let P be a poset. Then G_P is an interval graph if and only if P contains neither any of P_1, P_2, P_3 nor dual of P_2 as a $\lhd$-preserving subposet, see Fig. 10.*

As the set of forbidden $\lhd$-preserving subposets is the same both for the interval graphs and for chordal graphs. We obtain an alternative proof for the statement that a cover-incomparability graph is interval if and only if it is chordal. As a consequence of this theorem and Corollary 1 we get the following characterization of poset whose C-I graphs are unit interval graphs.

Theorem 9. *Let P be a poset. Then G_P is a unit interval graph if and only if P does not contain any of P_1, P_2, P_3, P_2^* (the dual of P_2), S_1, S_2, S_3 or S_2^* (the dual of S_2) as a $\lhd$-preserving subposet, see Fig. 10 and Fig. 2.*

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