

# Edge-sum distinguishing labeling

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**Abstract.** We study *edge-sum distinguishing labeling*, a type of labeling recently introduced by Tuza in [Zs. Tuza, *Electronic Notes in Discrete Mathematics* 60, (2017), 61-68] in context of labeling games.

An *ESD labeling* of an  $n$ -vertex graph  $G$  is an injective mapping of integers 1 to  $l$  to its vertices such that for every edge, the sum of the integers on its endpoints is unique. If  $l$  equals to  $n$ , we speak about a *canonical ESD labeling*.

We focus primarily on structural properties of this labeling and show for several classes of graphs if they have or do not have a canonical ESD labeling. As an application we show some implications of these results for games based on ESD labeling. We also observe that ESD labeling is closely connected to the well-known notion of the *Sidon sequence* and *harmonious labeling*.

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## 1 Introduction and preliminaries

Graph labeling is a vivid area of combinatorics which started in the middle of 1960's. Much of the area is based on results of Rosa [7] and of Graham and Sloane [2]. Since then, over 200 different labelings were introduced. We refer to Gallian's survey [1], citing over 2500 papers, gathering most of the results in the area. Applications of labeling are both theoretical (Rosa introduced so-called *graceful* labelings to attack Ringel's conjecture on certain graph decompositions) and practical (for example the frequency assignment problem [4,10,5]).

We study *edge-sum distinguishing* (abbreviated as ESD) labeling, introduced by Tuza [9] in 2017. Tuza's primarily concern was to study several combinatorial games connected to this labeling. Our main objective is to study structural properties of this labeling on its own. However, as our secondary objective, we also give some results on game variants of edge-sum distinguishing labeling.

**Structure of the paper.** In the rest of this section we review basic definitions and show a broader context of ESD labeling to other existing notions in combinatorics. The second section deals with structural properties of ESD labeling. For various well-known classes of graphs we show if they have a canonical ESD labeling or not. In the third section we are concerned with game variants, the original motivation of Tuza. Finally, in the last section we summarize our results and propose some open problems.

**Notation.** We use the notation of West [11]. All graphs in the paper are finite, undirected, connected and without multiple edges, unless we say otherwise.

## 1.1 Basic definitions

We need to formally define what graph labeling is. We will need vertex labelings only.

**Definition 1.** Let  $G = (V, E)$  be a graph and let  $L \subseteq \mathbb{N}$  be a set of labels. Then a mapping  $\phi : V \rightarrow L$  is called a vertex labeling. We further say that vertex labeling is canonical if  $|V| = |L|$ .

We will often refer to edge-weights, induced by a vertex labeling.

**Definition 2.** Let  $G = (V, E)$  be a graph and  $\phi$  a vertex labeling on  $G$ . If both ends of an edge  $uv \in E$  are already labeled then the weight of the edge is defined as  $w_\phi(uv) = \phi(u) + \phi(v)$ .

Now we can finally introduce a definition of *edge-sum distinguishing labeling*.

**Definition 3.** Let  $G = (V, E)$  be a graph and  $L = \{1, \dots, l\}$ . Vertex labeling  $\phi : V \rightarrow L$  is called edge-sum distinguishing labeling (ESD labeling) if  $\phi$  is injective and holds

$$\forall e, f \in E : e \neq f \rightarrow w_\phi(e) \neq w_\phi(f).$$

We note that no ESD labeling exists in case  $|L| < |V|$ . We call a special case when  $|L| = |V|$  a *canonical ESD labeling*.

*Example 1.* Consider a path  $P_n$  and denote its vertices consecutively  $v_1, \dots, v_n$ . Choose a labeling  $\phi(v_i) = i$ . Clearly, this is an ESD labeling and even a canonical ESD labeling.

## 1.2 Connections to existing notions

**Sidon sequences.** The Sidon sequences were introduced by Simon Sidon in 1932 [8]. We refer the reader to a dynamically updated survey of O’Bryant [6]. The formulation of the following definition comes from that survey.

**Definition 4.** A Sidon sequence is a sequence of integers  $a_1 < a_2 < \dots$  with the property that sums  $a_i + a_j$  ( $i \leq j$ ), are distinct.

ESD labeling can be reformulated in a similar fashion.

**Definition 5.** An ESD labeling of a graph  $G = (V, E)$ ,  $V = \{1, \dots, n\}$ , is a sequence of integers  $a_1 < a_2 < \dots$  with the property that sums  $a_i + a_j$ ,  $i \leq j$ ,  $(i, j) \in E$ , are distinct and  $a_1 = 1$ .

In the light of this new definition, we see that ESD labeling is in some sense a generalization of the Sidon sequence. The difference that  $a_1 = 1$  in the definition of ESD labeling could be easily dropped (but it is convenient for this paper). Also, one can observe that without this condition, the original Sidon sequences are ESD labelings of a sufficiently large complete graph with loops added to each vertex. However, again for our convenience, we consider only loopless graphs in this paper.

**Harmonious labeling.** *Harmonious labeling* was introduced by Graham and Sloane [2]. We say that graph  $G$  with  $k$  edges is *harmonious* if its vertices can be labeled injectively with integers modulo  $k$  so that the sum of the labels of its endpoints modulo  $k$  is unique.

The difference between harmonious and ESD labeling is that we do not take vertex labels and edge labels modulo number of edges in ESD case. In fact, ESD labelings and harmonious labeling behave differently. For example, it is conjectured that trees are harmonious and it is known that not all cycles are harmonious [3]. For comparison, we show in this paper that all trees and cycles have a canonical ESD labeling.

## 2 Structural results

### 2.1 Necessary condition

**Theorem 1.** If a graph  $G = (V, E)$  such that  $|V| > 1$  has a canonical ESD labeling, then the inequality  $|E| \leq 2|V| - 3$  holds.

*Proof.* We claim that every canonical ESD labeling of an  $n$ -vertex graph has at most  $2n - 3$  different edge-weights.

To prove this, observe that the smallest possible edge-weight in such labeling is 3 and the largest possible is  $2n - 1$ . Also, the edge-weights of  $G$  form a subset of the set  $\{3, \dots, 2n - 1\}$  which is of the size  $2n - 3$ . This proves the claim.

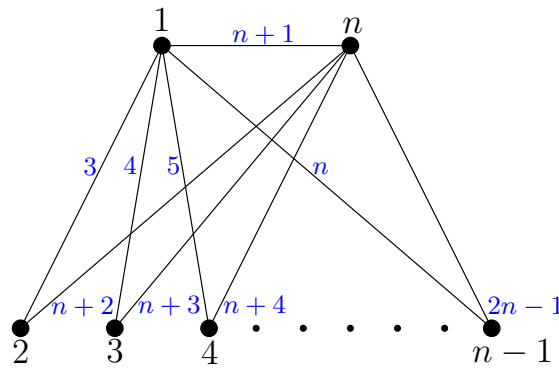
Now if a graph  $G$  has more than  $2|V| - 3$  edges we can use our claim and by the pigeonhole principle, we have two edges with the same weight, a contradiction.  $\square$

Now we will show that this bound is tight.

**Theorem 2.** *For every  $n \in \mathbb{N}, n > 1$ , there exist an  $n$ -vertex graph  $G_n$  with  $|E(G_n)| = 2n - 3$  which has a canonical ESD labeling.*

*Proof.* For  $G_2$  take  $K_2$  and for  $G_3$  take  $K_3$ . These cases are trivial.

For  $n > 3$ , take a complete bipartite graph  $K_{2, n-2}$  and add an edge between the two vertices of the part of size 2. See Figure 1 for an example.



**Fig. 1:** An example of an ESD graph with  $2n - 3$  edges.

We will show that this graph has a canonical ESD labeling. We will denote  $x_1, x_2$  the vertices of the part of size 2 and  $y_1, \dots, y_{n-2}$  the vertices of the other part.

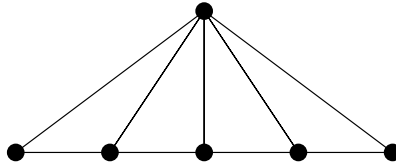
Now we define a labeling  $\phi$  in the following way.

- Let  $\phi(x_1) = 1$  and  $\phi(x_2) = n$ .
- Let  $\phi(y_i) = i + 1$  for  $1 \leq i \leq n - 2$ .

Observe that the edges incident with  $x_1$  have edge-weights from 3 to  $n + 1$ . Furthermore, the edges incident with  $x_2$ , except for the edge  $x_1x_2$ , have edge-weights ranging from  $n + 2$  to  $2n - 1$ . All these weights appear exactly once and thus we are done.  $\square$

**Fan graphs.** We showed a necessary condition for graph to have a canonical ESD labeling. The point of this part is to show that this condition is not sufficient in general by proving that *fan graphs*, which have  $2n - 3$  edges, do not have a canonical ESD labeling if their order is bigger than 8.

**Definition 6.** A fan graph  $F_n$  is a path  $P_{n-1}$  and one other vertex  $v$  (we call it the central vertex) joined by an edge with every vertex of the path. See Figure 2 for an example.



**Fig. 2:** A fan graph  $F_6$ .

**Theorem 3.** For every  $n \geq 8$  a fan graph  $F_n$  does not have a canonical ESD labeling.

*Proof.* From Theorem 1 we know that we have at most  $2n - 3$  different edge-weights. Since a fan graph of order  $n$  has exactly  $2n - 3$  edges we need to use every possible edge-weight from the set  $\{3, \dots, 2n - 1\}$  exactly once.

The edge-weights 3 and 4 can be obtained in exactly one possible way. In the first case on an edge with endpoints labeled 1 and 2, in the second case on an edge with endpoints 1 and 3. The edge-weight 5 can be obtained in two ways. Either as the weight of an edge with endpoints 2 and 3 or as the weight on an edge with endpoints 1 and 4. We get two possible subgraphs  $S_1$  and  $S_2$ .

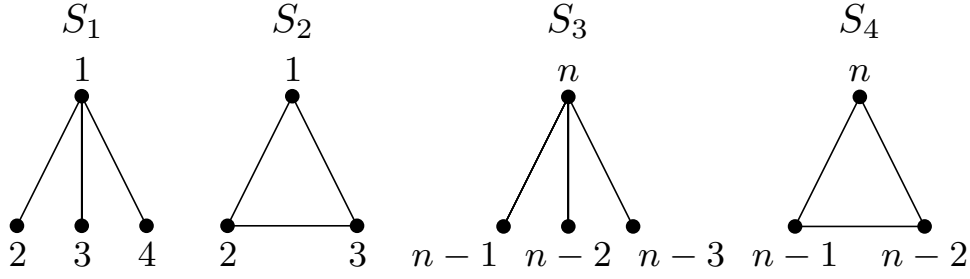
By a similar analysis, one can get the labeled subgraphs  $S_3$  and  $S_4$ .

Hence, exactly one of the labeled subgraphs  $S_1$  or  $S_2$  has to be in  $F_n$  and, analogously, one of the  $S_3$  and  $S_4$  as well. However, in all graphs  $S_i, i \in \{1, \dots, 4\}$ , one of its vertices has to be the central vertex. Since  $n \geq 8$ , we see that the minimum possible label in  $S_3$  and  $S_4$  is 5. Also, the maximum label on  $S_1$  and  $S_2$  is 4. Therefore, we cannot properly label the central vertex and the theorem follows.  $\square$

We note that  $F_n$  for  $n$  up to 7 has a canonical ESD labeling.

## 2.2 Complete bipartite graphs

We need to introduce a notion of isomorphism for vertex labelings.



**Fig. 3:** The subgraphs from the proof of Theorem 3.

**Definition 7.** Vertex labelings  $\phi_1$  and  $\phi_2$  on  $G$  are isomorphic if there exists an automorphism  $f$  of  $G$  such that  $\phi_1(v) = \phi_2(f(v))$  for every  $v \in V(G)$ .

We will prove the following theorem, covering all cases for complete bipartite graphs.

**Theorem 4.** Let  $K_{p,q}$  be a complete bipartite graph,  $p \leq q$ , then the following holds.

1. For  $p, q > 2$  there is no ESD labeling on  $K_{p,q}$ .
2. If  $p = 2$ , then there exists exactly one possible ESD labeling up to isomorphism.
3. If  $p = 1$ , then every canonical labeling is an ESD labeling.

*Proof.* 1. Suppose for a contradiction that we have some canonical ESD labeling  $\phi$ . Denote the parts of  $K_{p,q}$  by  $P$  and  $Q$ . We will divide the proof into two cases.

- There exist two vertices  $v_1, v_2$ , in  $P$  and two vertices  $u_1, u_2$  in  $Q$  such that  $\phi(v_2) = \phi(v_1) + 1$  and  $\phi(u_2) = \phi(u_1) + 1$ . Then  $w_\phi(v_1u_2) = w_\phi(v_2u_1)$ , and we get a contradiction.
  - There exist a part (without loss of generality  $P$ ) such that  $\phi(p_1) \neq \phi(p_2) + 1$  for every  $p_1, p_2 \in P$ . Since  $P$  is of size at least 3, there exist two vertices  $s_1, s_2 \in P$  with labels smaller than  $n$ . Thus there exists a vertex  $t_1 \in Q$  with label  $\phi(s_1) + 1$  and  $t_2 \in Q$  with label  $\phi(s_2) + 1$ . Then  $w_\phi(s_1t_2) = w_\phi(s_2t_1)$ , a contradiction.
2. We denote the vertices of the part of the size 2 as  $v_1, v_2$ . The vertices of the other part will be  $u_1, \dots, u_q$ . Let  $\psi$  be a vertex labeling of  $K_{2,q}$  defined as follows:
- $\psi(v_1) = 1$ ,
  - $\psi(v_2) = n$ ,
  - $\psi(u_i) = i + 1$  for  $i \in \{1, \dots, q\}$ .

Observe that  $\psi$  is indeed a canonical ESD labeling. For  $q = 2$ , one can easily check that this is the only canonical ESD labeling up to isomorphism.

Now, for a contradiction, assume that a canonical ESD labeling  $\psi'$ , non-isomorphic to  $\psi$ , exists. Furthermore,  $n > 4$ , and we can assume that  $\psi'(v_1) < \psi'(v_2)$ . Either  $\psi'(v_1) \neq 1$  or  $\psi'(v_2) \neq n$ . We distinguish two cases.

(a) It holds that  $\psi'(v_2) = \psi'(v_1) + 1$ .

Since  $n > 4$ , we can find two vertices  $a_1, a_2$  in the other part such that  $\psi'(a_2) = \psi'(a_1) + 1$ . Similarly as in case (1) of this theorem,  $w_{\psi'}(v_1 a_2) = w_{\psi'}(v_2 a_1)$  and we get a contradiction.

(b) It holds that  $\psi'(v_2) \neq \psi'(v_1) + 1$ .

Then there exist two distinct vertices  $u_j, u_k \in \{u_1, \dots, u_q\}$  such that one of the following holds. Either  $\psi'(u_j) = \psi'(v_1) + 1$  and  $\psi'(u_k) = \psi'(v_2) + 1$ , or  $\psi'(u_j) = \psi'(v_1) - 1$  and  $\psi'(u_k) = \psi'(v_2) - 1$ . In both cases  $w_{\psi'}(v_1 u_k) = w_{\psi'}(v_2 u_j)$  and we are done.

We conclude that no such  $\psi'$  exists.

3. Every edge in a canonical labeling of  $K_{1,q}$  has a unique sum since every edge is incident to the central vertex of degree  $q$ .

□

We note that the first part of Theorem 4 can be proved by using Theorem 1 but we think that our proof is more clear.

## 2.3 Trees

We already showed that paths and stars are ESD graphs. The following theorem solves the general case of trees.

**Theorem 5.** *Every tree has a canonical ESD labeling.*

*Proof.* Let  $T$  be an  $n$ -vertex tree with root in  $v_1 \in V(T)$ . We will denote by  $v_1, \dots, v_n$  an ordering of vertices visited in a breadth-first search on  $T$ , starting in  $v_1$ . We define a labeling  $\phi$  as  $\phi(v_k) := k, \forall v_k \in V(T)$ . We want to show that  $\phi$  is a canonical ESD labeling.

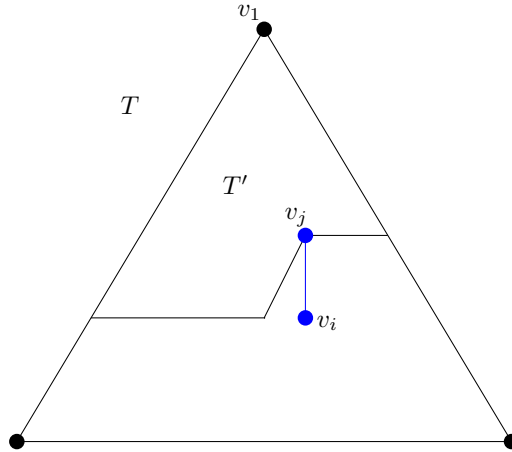
Consider some vertex  $v_i, i > 1$ , and its parent  $v_j$ . Denote by  $T'$  the tree induced by vertices  $v_1, \dots, v_{i-1}$ . See Figure 4 for an illustration. We claim that the following holds:

$$w_\phi(v_i v_j) > w_\phi(v_a v_b), \forall v_a v_b \in E(T').$$

By the level of a vertex we mean its distance to root vertex  $v_1$ . Without loss of generality, assume that  $a < b$ . We distinguish these cases.

- The edge  $v_a v_b$  has both endpoints on a level lower or equal to the level of  $v_j$ . Then  $a < j$  and  $b < i$  and from that  $a + b < i + j$ .
- If  $v_a = v_j$ , then  $v_j$  is the common parent of  $v_b$  and  $v_i$ . Thus  $b < i$  and from that  $b + j < i + j$ .
- The vertex  $v_a$  is on the same level as  $v_j$  and  $v_a \neq v_j$ . Then  $a < j$  and  $b < i$ , implying that  $a + b < i + j$ .

We proved the claim and the theorem follows.  $\square$



**Fig. 4:** An illustration of situation in Theorem 5.

## 2.4 Cycles

**Theorem 6.** *Every cycle graph  $C_n$  is an ESD graph.*

*Proof.* Let us denote the vertices of  $C_n$  as  $v_1, \dots, v_n$  in a circular order. We distinguish two cases:

1. If  $n$  is even, then we assign labels as follows:

- $\phi(v_i) = i$  for all  $i \in \{1, \dots, n-2\}$ ,
- $\phi(v_{n-1}) = n$ ,
- $\phi(v_n) = n-1$ .

Weights of the edges  $v_i v_{i+1}$  for  $i \in \{1, \dots, n-3\}$  are odd integers  $3, 5, \dots, 2n-5$ . The weight of the edge  $v_{n-1} v_n$  is  $2n-1$  and therefore is odd as well. The remaining edges will be even;  $w_\phi(v_n v_1) = n$  and  $w_\phi(v_{n-2} v_{n-1}) = 2n-2$ . We conclude that the edge-weights are unique.

2. If  $n$  is odd we assign labels as follows:

- $\phi(v_i) = i$  for all  $i \in \{1, \dots, n\}$ .



The weights of the edges between  $v_i v_{i+1}$  for  $i \in \{1, \dots, n-1\}$  will be odd integers  $3, 5, \dots, 2n-1$ . The weight of the edge  $v_1 v_n$  is equal to  $n+1$  and therefore it is even. Again, all edge-weights are unique and we get a canonical ESD labeling.

□

## 2.5 Generalized sunlet graphs

We recall that a graph is *unicyclic* if it contains exactly one cycle.

**Definition 8.** A generalized sunlet graph  $S_k^p$  is a unicyclic graph obtained by taking a cycle graph  $C_k$ , with  $V(C_k) = \{c_1, \dots, c_k\}$ , and joining path graphs  $R_i, i \in \{1, \dots, k\}$  of order  $p$  to this cycle so that one of the endpoints of  $R_i$  is identified with  $c_i$ .

**Theorem 7.** Let  $S_k^p$  be a generalized sunlet graph. If  $k$  is odd and  $p$  is even, then  $G$  has a canonical ESD labeling.

*Proof.* We denote the vertices of  $S_k^p$  in the following way.

- Vertices on the cycle are  $v_1, v_{p+1}, v_{2p+1}, \dots, v_{(k-1)p+1}$ .
- Vertices on the path joined to the vertex  $v_{ip+1}$  are consecutively  $v_{ip+1}, \dots, v_{(i+1)p}$ , for  $1 \leq i \leq k$ .

We define a vertex labeling  $\phi$  as  $\phi(v_i) := i$ . We claim that  $\phi$  is a canonical ESD labeling. All edge-weights on attached paths are odd, because we get them as a sum of two consecutive numbers. Furthermore, all edge-weights on a path joined to vertex  $v_{ip+1}$  are smaller than edge-weights on a path joined to vertex  $v_{(i+1)p+1}$ . Thus all edge-weights on paths are distinct. All edge-weights on the cycle except for the edge  $v_1 v_{(k-1)p+1}$  are in the form  $k'p+2$  for some  $k' \in \mathbb{N}$ . Thus they are all even and distinct.

It remains to show that the edge  $v_1 v_{(k-1)p+1}$  has an edge-weight different from all others. For a contradiction we assume that the edge-weight  $(k-1)p+2$  was already used. It is even, so it can be only used on the cycle. Thus,  $k-1$  must be a sum of two distinct consecutive natural numbers. That gives a contradiction, because  $k-1$  is even. □

## 2.6 Grids

**Definition 9.** An  $k \times l$  grid graph  $G_{k,l}$  is the Cartesian product of path graphs  $P_k$  and  $P_l$ .

**Theorem 8.** Let  $G_{k,l}$  be a grid graph. If  $k$  or  $l$  is even then  $G_{k,l}$  has a canonical ESD labeling.

*Proof.* Without loss of generality assume that  $k$ , the number of columns, is even. Let us denote the vertices in the  $i$ -th row by  $v_{(i-1)k+1}, \dots, v_{ik}$  for every  $i \in \{1, \dots, l\}$ . We define a canonical vertex labeling  $\phi$  as  $\phi(v_i) := i$ . We want to show that  $\phi$  is an ESD labeling on  $G_{k,l}$ .

The graph  $G_{k,l}$  with labeling  $\phi$  has the following edge-weights:

- $2(i-1)k + 3, \dots, 2ik - 1$  in the  $i$ -th row for every  $1 \leq i \leq l$ ,
- $2ik + 3, \dots, 2(i+1)k - 1$  in the  $(i+1)$ -th row for every  $0 \leq i \leq l-1$ ,
- $(2i-1)k + 2, \dots, (2i+1)k$  on edges between the  $i$ -th and the  $(i+1)$ -th row  $1 \leq i \leq l-1$ .

All edge-weights on rows are odd and all edge-weights in the  $i$ -th row are smaller than all edge-weights in the  $(i+1)$ -th row. A similar argument holds for all edge-weights in columns. This concludes the proof.  $\square$

## 2.7 Complete graphs

From Theorem 1 it is clear that complete graphs  $K_n$  for  $n > 3$  do not have a canonical ESD labeling.

We note that the problem of finding  $l$  sufficiently large so that we have a ESD labeling with  $l$  labels on complete graphs is connected to finding precise bounds on the shortest Sidon sequence problem. The survey of O'Bryant [6] features a table, reproduced here in Figure 5, giving upper bounds on how big  $l$  must be for a graph  $K_n$ , with  $n$  up to 13. This is captured in Table 1.

$n$	2	3	4	5	6	7	8	9	10	11	12	13
min. $l$	1	3	$\leq 6$	$\leq 11$	$\leq 17$	$\leq 25$	$\leq 34$	$\leq 44$	$\leq 55$	$\leq 72$	$\leq 85$	$\leq 106$

**Table 1:** A table showing bounds on the minimum  $l$  such that  $K_n$  has an ESD labeling with  $l$  labels.

## 3 Games with ESD labeling

Tuza in his paper [9] emphasized that only few papers on graph labeling games exist. He defined a new game from ESD labeling.

**Definition 10.** *We call a vertex of graph free if it is not labeled yet.*

**Definition 11.** Let  $G = (V, E)$  be a graph and  $L = \{1, \dots, l\}$  its set of labels. Alice and Bob are two players who alternate after every move. Alice starts. In each move, player chooses a free vertex of  $G$  and assigns an unused label to it. The move is legal if the resulting edge-weights are unique.

The game ends if there is no legal move possible or an ESD labeling is created. Alice wins if an ESD labeling is created, otherwise Bob wins.

We say that an ESD labeling game is canonical on  $G$  if  $|L| = |V(G)|$ .

One can also define other variants of this game. For example, Bob can be the starting one. Also, our definition of game is a Maker-Breaker type of game, but it is possible to define Achievement and Avoidance type of this game as well.

**Proposition 1.** If a graph  $G$  does not have a canonical ESD labeling then Bob has a winning strategy in the canonical game on  $G$ .

*Proof.* If a graph  $G$  does not have a canonical ESD labeling then Alice can not make any canonical ESD labeling and Bob eventually wins.  $\square$

**Theorem 9.** Alice wins every canonical game on a star  $S_n$ .

*Proof.* We already proved in Theorem 4 that every canonical vertex labeling on a star graph is edge-sum distinguishing. Thus Alice wins every game regardless on the course of the game.  $\square$

**Theorem 10.** Bob wins every canonical game on a complete bipartite graph  $K_{p,q}$ ,  $p \leq q$ , where  $p = 2$ .

*Proof.* We recall Theorem 4. The graph  $K_{p,q}$ ,  $p \leq q$ , where  $p = 2$ , needs to have labels 1 and  $p + q$  on the smaller part. Thus a winning strategy for Bob is to assign a label  $w$ , such that  $1 < w < p + q$ , on a free vertex of the smaller part in his first move. Now it is not possible to build a canonical ESD labeling and Bob wins.  $\square$

Tuza also asked [9, Problem 3.1] the following question: Given  $G = (V, E)$ , for which values of  $l$  can Alice win the edge-sum distinguishing labeling game? We partially answer this question by the following theorem.

**Theorem 11.** Let  $G$  be a graph,  $\Delta$  its maximum degree, and  $L$  its set of labels. If  $|L| \geq (\Delta^2 + 1)n + \Delta \binom{n-1}{2}$ , then Alice has a winning strategy.

*Proof.* For each vertex  $v$  of  $G$ , define a set  $S_v$  as the set of labels available for  $v$ . In the beginning of every game,  $S_v = L$  for every  $v \in V(G)$ .

Our goal is to build a winning strategy for Alice. In  $k$ -th move, a player assigns to some free vertex  $v$  some label  $\phi(v) \in S_v$ . We update the set of labels in the following three steps right after the player's choice.

1. We delete  $\phi(v)$  from  $S_u$  for every  $u \in V(G)$ . This label cannot be used twice, since ESD labeling is an injective mapping.
2. For every free vertex  $y$  incident to  $v$  we delete all labels  $l_{y,e}$  such that  $l_{y,e} + \phi(v) = w_\phi(e)$  for some edge  $e$  with both endpoint vertices labeled and incident with  $v$ . In this process, we delete at most  $\binom{k-1}{2}$  labels from  $S_y$ .
3. For every free vertex  $z$  and for every vertex  $z' \in N(z)$ , such that  $z'$  is already labeled, we delete from  $S_z$  all labels  $l'$  such that

$$l' + \phi(z') = w_\phi(vv'), \quad \forall v' \in N(v).$$

Within these steps, we delete at most  $\Delta^2$  labels from label set of every free vertex.

If before every move is the label set for every free vertex nonempty, Alice wins. Let us count how many labels are deleted in course of the game for every free vertex.

- We delete at most  $n - 1$  labels through all first steps.
- We delete at most  $\Delta \binom{n-1}{2}$  labels through all second steps.
- We delete at most  $\Delta^2 n$  labels in third steps.

Summarized, we delete at most  $(\Delta^2 + 1)n + \Delta \binom{n-1}{2} - 1$  labels. If we have one extra label available, we can always find a label for a free vertex and our bound is proved. Note an important fact that it does not matter how Bob plays and the resulting labeling is ESD.  $\square$

Observe that this theorem also gives us a bound on the size of label set for general graphs. This follows by taking Proposition 1 into account.

Also, by a similar analysis, one can get the following theorem for path graphs.

**Theorem 12.** *Let  $P_n$  be a path graph on  $n$  vertices. If  $|L| \geq 5n$ , then Alice wins every game on  $P_n$ .*

## 4 Concluding remarks

We studied a new type of graph labeling, introduced by Tuza, which is similar to harmonious labeling and has a relation to the Sidon sequences. We would like to summarize our main results.

- We proved that trees, cycles and complete bipartite graphs with one part of size 2 have a canonical ESD labeling.
- We proved that in some cases grid graphs and generalized sunlet graphs do have a canonical ESD labeling.

- We showed that fan graphs and complete bipartite graphs with both parts of size at least 3 do not have a canonical ESD labeling.
- We studied a Maker-Breaker type of game, applied our previous results and derived a general bound on number of labels such that Maker wins the game.

**Open problems.** Aside from Tuza’s original problems proposed in [9], we emphasize the following question, arising from the results in this paper.

*Problem 1.* What is the maximum possible number of edges for  $n$ -vertex connected graphs so that every graph with such number of edges has a canonical ESD labeling?

From Theorem 5 we see that to answer this question one needs to resolve the case of unicyclic graphs which is now only partially solved.

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