Reconfiguration Graph for Vertex Colourings of Weakly Chordal Graphs

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Abstract

The reconfiguration graph $R_k(G)$ of the k-colourings of a graph G contains as its vertex set the k-colourings of G and two colourings are joined by an edge if they differ in colour on just one vertex of G.

We answer a question of Bonamy, Johnson, Lignos, Patel and Paulusma by constructing for each $k \geq 3$ a k-colourable weakly chordal graph G such that $R_{k+1}(G)$ is disconnected. We also introduce a subclass of k-colourable weakly chordal graphs which we call k-colour compact and show that for each k-colour compact graph G on n vertices, $R_{k+1}(G)$ has diameter $O(n^2)$. We show that this class contains all k-colourable co-chordal graphs and when $k = 3$ all 3-colourable $(P_5, \overline{P_5}, C_5)$-free graphs. We also mention some open problems.

1 Introduction

Let G be a graph, and let k be a non-negative integer. A k-colouring of G is a function $f : V(G) \to \{1, \dots, k\}$ such that $f(u) \neq f(v)$ whenever $(u, v) \in E(G)$. The reconfiguration graph $R_k(G)$ of the k-colourings of G has as vertex set the set of all k-colourings of G and two vertices of $R_k(G)$ are adjacent if they differ on the colour of exactly one vertex (the change of the colour is the so called *colour switch*). For a positive integer ℓ, the *ℓ-colour diameter* of a graph G is the diameter of $R_\ell(G)$.

In the area of reconfigurations for colourings of graphs, one focus is to determine the complexity of deciding whether two given colourings of a graph can be transformed into one another by a sequence of recolourings (that is, to decide whether there is a path between

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these two colourings in the reconfiguration graph); see, for example, [8, 7, 5, 3]. Another focus is to determine the diameter of the reconfiguration graph in case it is connected or the diameter of its components if it is disconnected [2, 6, 1, 4, 9]. We refer the reader to [13, 11] for excellent surveys on reconfiguration problems. In this note, we continue the latter line of study of reconfiguration problems. In Section 3, we shall answer in the negative a question of Bonamy, Johnson, Lignos, Patel and Paulusma [2] concerned with the $(k+1)$-colour diameter of k-colourable perfect graphs by showing that it can be infinite. On the positive side, in Section 4, we shall consider two specific subclasses of k-colourable perfect graphs and show that their $(k+1)$-colour diameter is quadratic in the order of the graph.

2 Preliminaries

For a graph $G = (V, E)$ and a vertex $u \in V$, let $N_G(u) = \{v : uv \in E\}$, A *separator* of a graph $G = (V, E)$ is a set $S \subset V$ such that $G - S$ has more connected components than G. If two vertices u and v that belong to the same connected component in G are in two different connected components of $G - S$, then we say that S *separates* u and v. A *chordless path* P_n of length $n-1$ is the graph with vertices $v_1, \ldots, v_n$ and edges v_iv_{i+1} for $i = 1, \ldots, n-1$. It is a *cycle* C_n of length n if the edge v_1v_n is also present.

The *complement* of G is denoted $\overline{G} = (V, \overline{E})$. It is the graph on the same vertex set as G and there is an edge in G between two vertices u and v if and only if there is no edge between u and v in $\overline{G}$. A set of vertices in a graph is *anticonnected* if it induces a graph whose complement is connected. A *clique* or a *complete graph* is a graph where every pair of vertices is joined by an edge. The size of a largest clique in a graph G is denoted $\omega(G)$. The *chromatic number* $\chi(G)$ of a graph G is the least integer k such that G is k-colourable.

A graph G is *perfect* if $\omega(G') = \chi(G')$ for every (not necessarily proper) subgraph G' of G. A *hole* in a graph is a cycle of length at least 5 and an *antihole* is the complement of a hole. A graph is *weakly chordal* if it is (hole, antihole)-free. A graph is *co-chordal* if it is $(\overline{C_4}$, anti-hole)-free. Every weakly chordal graph is perfect. Every co-chordal graph and $(P_5, \overline{P_5}, C_5)$-free graph is weakly chordal.

A *2-pair* of a graph G is a pair $\{x, y\}$ of nonadjacent distinct vertices of G such that every chordless path from x to y has length 2. We often use the following well-known lemma:

Lemma 2.1 (Hayward et al. [10]). *A graph G is weakly chordal graph if and only if every subgraph of G is either a complete graph or it contains a 2-pair.*

3 Weakly chordal graphs

In this section, we consider a question from [2] which asks whether the $(k+1)$-colour diameter of k-colourable perfect graphs is quadratic. We answer this question in the negative in the following theorem.

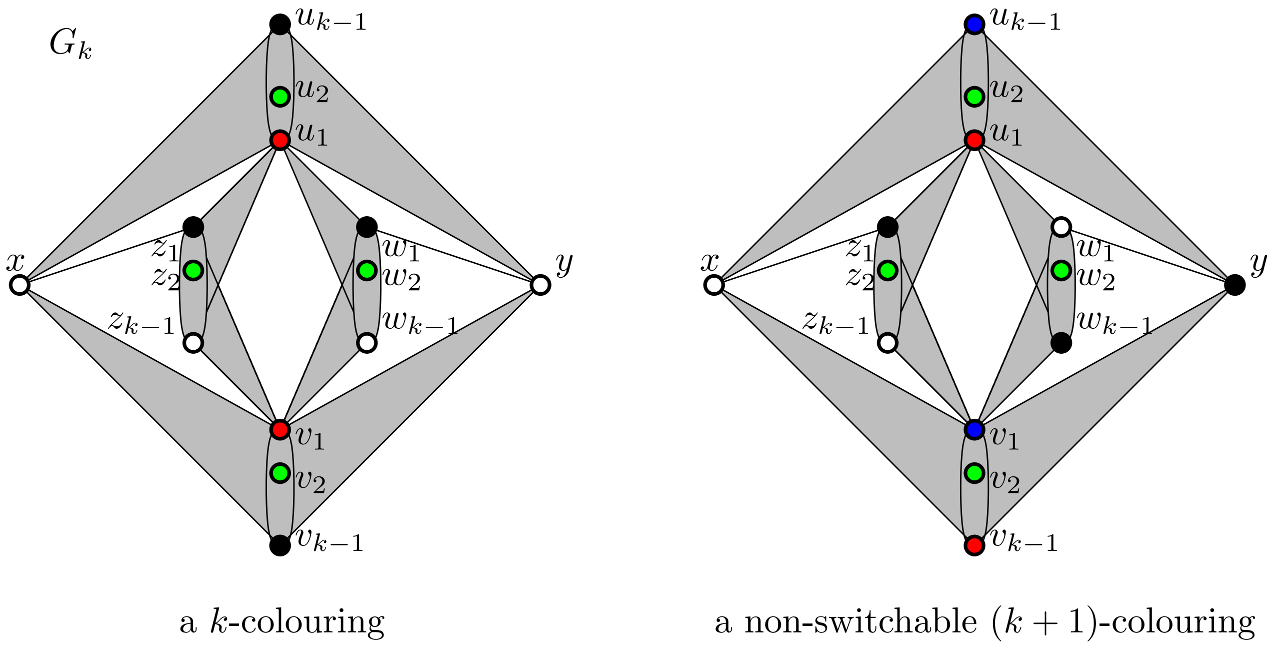

Figure 1: The graph G_k. Each gray area corresponds to a clique.

Theorem 3.1. *For each $k \geq 3$ there exists a k-colourable weakly chordal graph G such that $R_{k+1}(G)$ is disconnected.*

In other words, Theorem 3.1 states that for each $k \geq 3$ the $(k+1)$-colour diameter of k-colourable weakly chordal graphs can be infinite. Since the class of weakly chordal graphs is a proper subclass of the class of perfect graphs, Theorem 3.1 answers their question in a strong form. It is worth mentioning that the case $k = 2$ is already known [2] as the class of 2-colourable weakly chordal graphs is precisely the class of chordal bipartite graphs.

Proof of Theorem 3.1. It suffices to construct for each $k \geq 3$ a k-colourable weakly chordal graph G_k and a $(k+1)$-colouring of G such that each of the $k+1$ colours appear in the closed neighbourhood of every vertex of G_k, as then no vertex of G_k can get recoloured.

Such graph is depicted in Figure 1. It is formed from the disjoint union of four complete graphs K_{k-1}, one on vertices u_i for $i \in \{1, \dots, k-1\}$, the other three on vertices v_i, w_i, z_i, respectively, and two further vertices x and y. These parts are joined together by additional edges such that x and y are connected to each u_i and to each v_i; u_1 and v_1 are connected to each z_i and to each w_i; and finally, G_k contains two further edges xz_1 and yw_1.

A possible k-colouring of G_k is schematically shown on the left side of Figure 1; in both pictures each 4-tuple u_i, v_i, w_i, z_i for $i \in \{2, \dots, k-2\}$ receives a unique colour. On the other hand, in the $(k+1)$-colouring depicted on the right, every vertex of G_k has its neighbours coloured by the k remaining colours, hence no vertex can be recoloured. Hence, this colouring corresponds to an isolated vertex in the reconfiguration graph, and thus the reconfiguration graph $R_{k+1}(G_k)$ is disconnected.

It remains to show that G_k is weakly chordal. Observe first that any distinct u_i, u_j with $i, j \in \{2, \dots, k-1\}$ have the same neighbourhood, hence no hole and no antihole may contain both of them. The same holds for vertices v_i, w_i, z_i, respectively. Hence without loss of generality we may assume that no vertex with index at least three participates in a hole nor in an antihole. In other words, it suffices to restrict ourselves only to the graph G_3 to show that it is weakly chordal.

By examining possible paths, one can also realise that vertices x, y as well as u_1, v_1 form a 2-pair. Since no hole may contain a 2-pair, we may assume without loss of generality that a possible hole does not contain y and v_1. The vertex x separates v_2 and also the vertex u_1 separates w_1, w_2 from the rest in the graph $G_3 - \{y, v_1\}$, hence v_2, w_1, w_2 also do not belong to a hole. We were left with five vertices x, u_1, u_2, z_1, z_2 which do not induce a hole, hence G_3 does not contain a hole at all.

Now assume for a contradiction that G_3 contains an antihole on at least 6 vertices. The graph G_3 contains only 6 vertices of degree at least 4, hence the antihole contains exactly 6 vertices. The neighbourhood of u_2 induces a diamond (a K_4 minus an edge), hence it does not belong to the antihole as no diamond is an induced subgraph of $\overline{C_6}$. The same holds for v_2, z_2, w_2. We are left with vertices x, y, u_1, v_1, w_1, z_1 which do not induce an antihole in G_3, hence G_3 does not contain an antihole at all.

Since G_3 is weakly chordal, we have shown that G_k is also weakly chordal for each $k \geq 4$. □

4 Quadratic diameter

In this section, we introduce a subclass of k-colourable weakly chordal graphs that we call *k-colour compact graphs*. We show in Theorem 4.1 that for each k-colour compact graph G on n vertices the diameter of $R_{k+1}(G)$ is $O(n^2)$. We then show in Lemma 4.1 that k-colourable co-chordal graphs are k-colour compact and in Lemma 4.2 that 3-colourable $(P_5, \overline{P_5}, C_5)$-free graphs are 3-colour compact.

For a 2-pair $\{u, v\}$ of a weakly chordal graph G, let $S(u, v) = N_G(u) \cap N_G(v)$. Note that, by the definition of a 2-pair, $S(u, v)$ is a separator of G that separates u and v. Let C_v denote the component of $G \setminus S(u, v)$ that contains the vertex v.

Definition 4.1. For a positive integer k, a k-colourable weakly chordal graph G is said to be *k-colour compact* if every subgraph H of G either

(i) is a complete graph, or

(ii) contains a 2-pair $\{x, y\}$ such that $N_H(x) \subseteq N_H(y)$, or

(iii) contains a 2-pair $\{x, y\}$ such that $C_x \cup S(x, y)$ induces a clique on at most three vertices.

Theorem 4.1. *Let k be a positive integer, and let G be a k-colour compact graph on n vertices. Then $R_{k+1}(G)$ has diameter $O(n^2)$.*

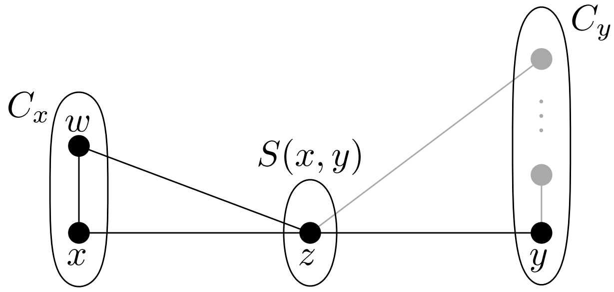

Figure 2: For the proof of Theorem 4.1. The component C_y contains further vertices and edges outlined in gray.

Proof. It suffices to show that we can recolour a $(k+1)$-colouring α of G to a $(k+1)$-colouring β by recolouring every vertex at most $2n$ times.

We first suppose that G is a complete graph. In this case, we know from [2] that we can recolour α to β by recolouring every vertex at most $2n$ times. We now consider the case when G is not a complete graph but satisfies condition (ii) of k-colour compact graphs. We use induction on the number of vertices of G. Let $\{x, y\}$ be a 2-pair of G such that $N_G(x) \subseteq N_G(y)$. From α and β, we can immediately recolour x with, respectively, $\alpha(y)$ and $\beta(y)$. Let $G' = G - \{x\}$, and let $\alpha_{G'}$ and $\beta_{G'}$ denote the restrictions of α and β to G'. By our induction hypothesis, we can transform $\alpha_{G'}$ to $\beta_{G'}$ by recolouring every vertex at most $2(n-1)$ times. We can extend this sequence of recolourings to a sequence of recolourings in G by recolouring x using the same recolouring as y. Then x gets recoloured as many times as y as needed.

Suppose that G satisfies condition (iii) of k-colour compact graphs. We use induction on the number of vertices. If $S(x, y)$ contains exactly two vertices, then $C_x = \{x\}$ and hence G satisfies condition (ii) of k-colour compact graphs. So we can assume that $S(x, y)$ is a single vertex z and C_x consists of x and another vertex w, see Figure 2. From α and β, we can recolour x and w to another colour either immediately or by first recolouring w and x, respectively. Let $G^* = G - \{x, w\}$. By our induction hypothesis, we can transform α_{G^*} to β_{G^*} by recolouring every vertex at most $2(n-2)$ times. We can extend this sequence of recolourings to a sequence of recolourings in G by recolouring x and w whenever z is recoloured to their colour. At the end of the sequence we recolour x and w so that they agree in both colourings. As x and w are recoloured at most two more times as z, this completes the proof. □

Lemma 4.1. *Every k-colourable co-chordal graph is k-colour compact.*

Proof. Let G be a k-colourable co-chordal graph. If G is a complete graph, then G is k-colour compact by definition. Otherwise, since G is weakly chordal, G contains a 2-pair $\{x, y\}$ by Lemma 2.1. If x has a neighbour x_1 that is not a neighbour of y and y has a neighbour y_1 that is not a neighbour of x, then x_1 is not adjacent to y_1, as otherwise $S(x, y)$ does not separate x and y. But then the edges xx_1 and yy_1 form $\overline{C_4}$, a contradiction. Therefore, $N_G(x) \subseteq N_G(y)$ or vice-versa and hence the graph G is k-colour compact as required. □

Lemma 4.2. *Every 3-colourable* $(P_5, \overline{P_5}, C_5)$*-free graph is 3-colour compact.*

The proof of this lemma will require a little more work. First, we need some definitions and auxiliary results. When T is a set of vertices of a graph G, a set $D \subseteq V(G) \setminus T$ is T-complete if each vertex of D is adjacent to each vertex of T. Let $D(T)$ denote the set of all T-complete vertices.

Lemma 4.3 (Trotignon and Vušković [12])**.** *Let G be a weakly chordal graph, and let $T \subseteq V(G)$ be a set of vertices such that $G[T]$ is anticonnected and $D(T)$ contains at least two non-adjacent vertices. If T is inclusion-wise maximal with respect to these properties, then any chordless path of $G \setminus T$ whose ends are in $D(T)$ has all its vertices in $D(T)$.*

The following corollary is implicit in [12].

Corollary 4.1. *Let G be a weakly chordal graph that contains a chordless path P of length 2. Then there exists an anticonnected set T containing the centre of P, such that $D(T)$ contains a 2-pair of G.*

In particular, the 2-pair can always be found in the neighbourhood of the centre of P.

Proof. We start with the centre of P to build our set T as in Lemma 4.3. Then $D(T)$ is not a clique as it contains both ends of P. Hence, by definition of weakly chordal graphs, $D(T)$ contains a 2-pair. This 2-pair is also a 2-pair of G by Lemma 4.3. □

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. Let G be a 3-colourable $(P_5, \overline{P_5}, C_5)$-free graph. Suppose towards a contradiction that G is not 3-colour compact. In particular, G is not a complete graph, as otherwise G would be 3-colour compact by the definition. Since G is weakly chordal, it contains a 2-pair $\{x, y\}$. We choose $\{x, y\}$ such that $|V(C_x)|$ is minimum over all 2-pairs $\{x, y\}$ of G.

Denote by G' the subgraph of G induced by the union of $S(x, y)$ and the vertices of C_x. The subgraph G' is not complete, as G would be 3-colour compact, hence G' contains a chordless path of length 2. Let us argue that G' contains a chordless path of length 2 whose centre is, in fact, a member of C_x.

If C_x is not complete, then this is immediate. And if $S(x, y)$ is not complete, then it contains a pair of vertices u and v that are not adjacent, so we take u, x, v to be our path. Hence we can assume that C_x and $S(x, y)$ are both complete and, as G' is not complete, there must be a vertex u of C_x and a vertex v of $S(x, y)$ such that $uv \notin E(G)$. Then we can take u, x, v as our path and our aim is achieved.

Applying Corollary 4.1 with P being a chordless path of length 2 whose centre is in C_x, we find that G' contains a 2-pair $\{z, w\}$ that is also a 2-pair of G.

We next want to argue that $z, w \in S(x, y)$. For a contradiction, assume without loss of generality that z belongs to C_x. This implies that $S(z, w) \subseteq V(G')$. So there must be a component C_1 of $G \setminus S(z, w)$ such that C_1 and $S(x, y)$ do not have a vertex in common since y is adjacent to every vertex of $S(x, y)$. Therefore, we find that $C_1 \subseteq C_x$. If $C_1 = C_x$,

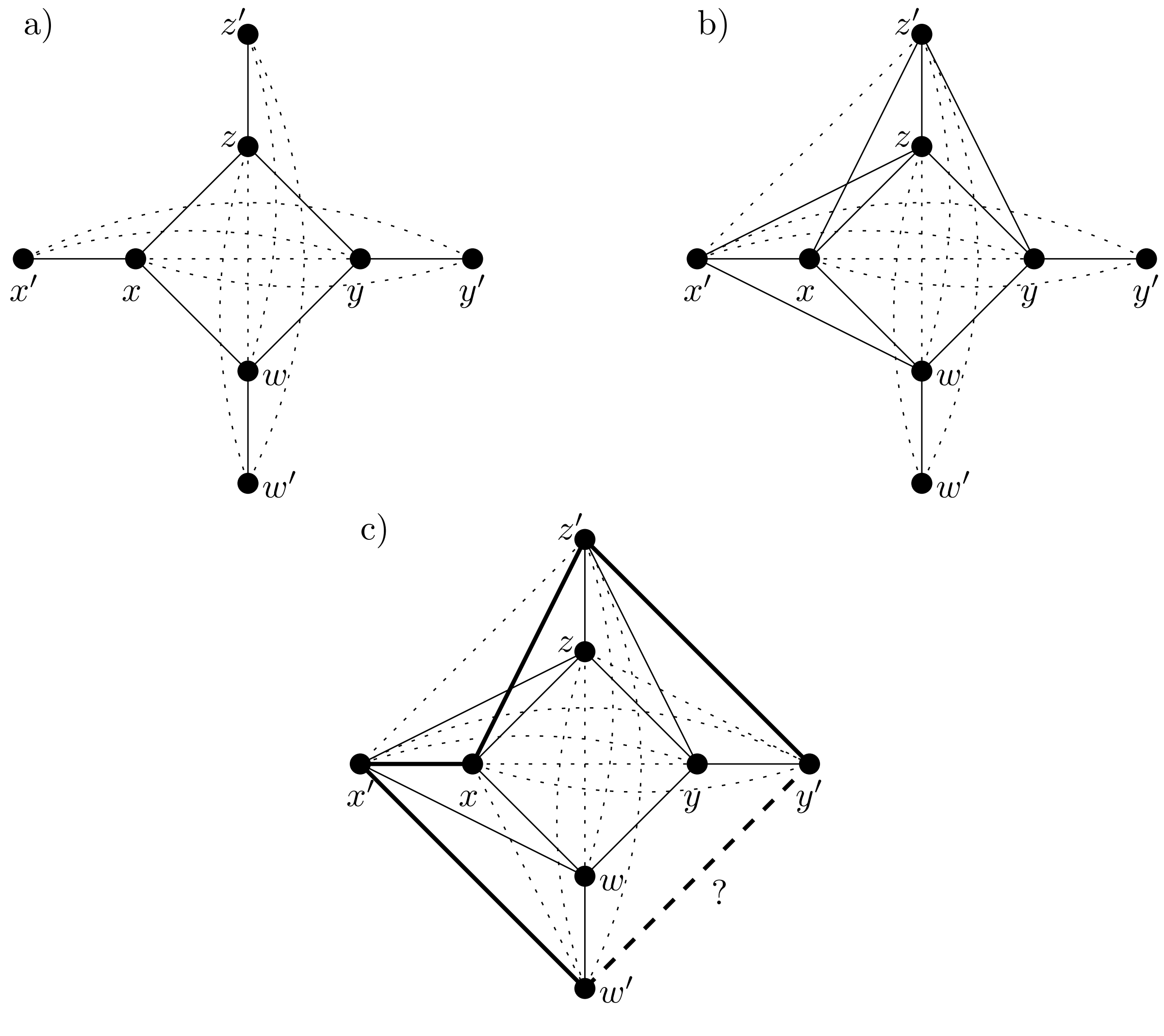

Figure 3: The case analysis for the proof of Lemma 4.2. The dotted connections indicate nonadjacent vertices.

then $S(x,y) = S(z,w)$ and hence $z, w \in C_1$ which is impossible because $S(z,w)$ separates z and w. Therefore, $|V(C_1)| < |V(C_x)|$ holds, which contradicts our choice of C_x.

Hence we have concluded that $z, w \in S(x,y)$. Now the vertices x, z, y, w form a cycle such that $\{x, y\}$ and $\{z, w\}$ are 2-pairs. Since G is not 3-colour compact, $N_G(x) \nsubseteq N_G(y)$, hence there exists a vertex x' that is adjacent to x but not to y. Analogously, there are vertices y', z', w' such that $yy', zz', ww' \in E$, but $xy', wz', zw' \notin E$. If $x' = z'$, then x' must be adjacent to y or w else x', x, z, y, w form $\overline{P_5}$. But x' is not adjacent to y and z' not adjacent to w, thus $x' \neq z'$. Similarly, $y' \neq w'$ and hence x', y', w', z' are distinct. Moreover, since $S(x,y)$ is a separator, we get $x'y' \notin E$ and analogously $w'z' \notin E$, see Figure 3 a).

If both z' and w' are not adjacent to x, then z', z, x, w, w' form P_5. So we can assume without loss of generality that z' is adjacent to x. If z' is not adjacent to y, then z', z, x, w, y would form $\overline{P_5}$ as w is not adjacent to z'. Thus z' is adjacent to both x and y.

Similarly, to avoid P_5 on vertices x', x, z, y, y' either x' or y' must be adjacent to z and hence also to w, so we assume without loss of generality that x' is adjacent to z and w. If x' is adjacent to z', then x, z, z', x' form K_4, a contradiction with the assumption the G is 3-colourable, see Figure 3 b).

To avoid a P_5 on the vertices x', x, z', y, y', the vertices z' and y' are forced to be adjacent. Similarly $x'w' \in E$ as otherwise z', z, x', w, w' induce a P_5. To avoid a K_4 on the vertices z, z', y, y', the vertices z and y' need to be nonadjacent. Similarly $xw' \notin E$ as otherwise x, x', w, w' induce a K_4.

Now the vertices w', x', x, z', y' induce either a C_5 or a P_5, depending whether the edge $y'w'$ is present or not, see Figure 3 c). In either case we arrive at a contradiction and the lemma is proved. □

We are aware the the concept of k-colour compact graphs does not fit tight with the class of $(P_5, \overline{P_5}, C_5)$-free graphs, as some of these graphs need not to be k-colour compact for $k \geq 4$. An example of such graph H for $k = 4$ is depicted in Figure. 4.

Due to symmetries of the graph H it suffices without loss of generality to consider only the 2-pair $\{x, y\}$ as other 2-pairs could be mapped onto $\{x, y\}$ by an automorphism of H. Observe that this 2-pair violates the conditions of the definition 4.1 for H to be 4-colour compact.

Any choice of five vertices from H would contain two vertices joined by a horizontal or a vertical edge, and such edge cannot be extended to an induced P_3, hence H is also P_5-free. Also, such choice of five vertices would contain two opposite vertices either of the inner C_4 or from the outer one, like the vertices x and y. As such two vertices form an 2-pair, H contains no C_5. Finally, H has only two induced C_4 and neither could be completed by any fifth vertex to a $\overline{P_5}$.

5 Concluding remarks

We end this note with three open problems.

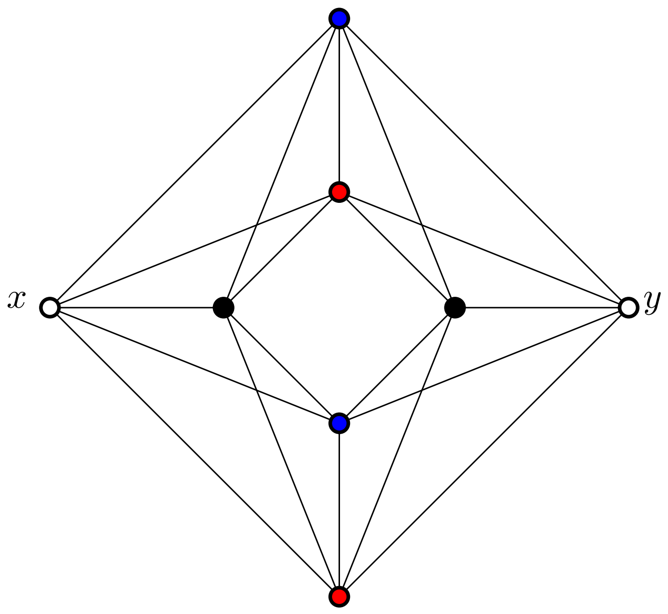

Figure 4: A $(P_5, \overline{P_5}, C_5)$-free 4-colourable graph H that is not 4-colour compact.

Problem 1. *For which integer $\ell > k+1$ is the ℓ-colour diameter of k-colourable weakly chordal graphs quadratic?*

Problem 2. *For which integer $\ell > k+1$ is the ℓ-colour diameter of k-colourable perfect graphs quadratic?*

Problem 3. *Is it true that the $(k+1)$-colour diameter of k-colourable $(P_5, \overline{P_5}, C_5)$-free graphs is quadratic for each $k \geq 4$?*

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