Ronald Lewis Graham

Just a Few Memories

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It was high summer of 1973, Keszthely, Hungary. An unusually large meeting "Finite and Infinite Sets" was held there in Hotel Helikon from June 25 till July 1, on the occasion of Paul Erdős 60'th birthday. It was an excellent meeting by any standards then and today too. It is instructive to page through its 3 volume proceedings [11]: totalling 1550 pages, containing papers by Rado, Tutte, de Bruijn, Straus, Berge, Galvin, Rudin, Guy, Selfridge, Hilton, McKenzie, Kleitman, Kunen, Milner, Neumann-Lara to name just a few. 12 papers coauthored by Erdős (including a joint paper with Lovász which inaugurated the Lovász Local Lemma), 3 papers by Shelah, 4 papers by Hajnal, 3 papers by Laver to list just a few contributions. And also 3 papers by Ron Graham all related to Ramsey with a total of more than 20 papers dealing with Ramsey type problems.

This was the meeting which for many years set high standards for universal combinatorial conferences which were held in 70s and 80s in France, Czechoslovakia, Hungary, Canada and elsewhere. It was the event of the year.

One of my strong memories of the meeting is a tall athletic man who excelled at everything. His name was known to me as well as some of his work (even in that pre-email and pre-internet age). But there he was: running, juggling, frisbeeing and showing tricks in everything from photography to handling magically an overhead projector (as far as I remember there wasn't a trampoline there). This was Ron Graham at his best, legendary already at that time. There we met for the first time.

My memory is vivid even now after years when in many meetings and collaboration I have seen that this youthful engaged style was Ron Graham's modus operandi. And later we all learned that many of these activities were not mere hobbies but professional level acts. What seemed to be easy and what Ron liked to display easily in his easygoing style was in fact hard learned and hard core. I believe this was symptomatic of his mathematics

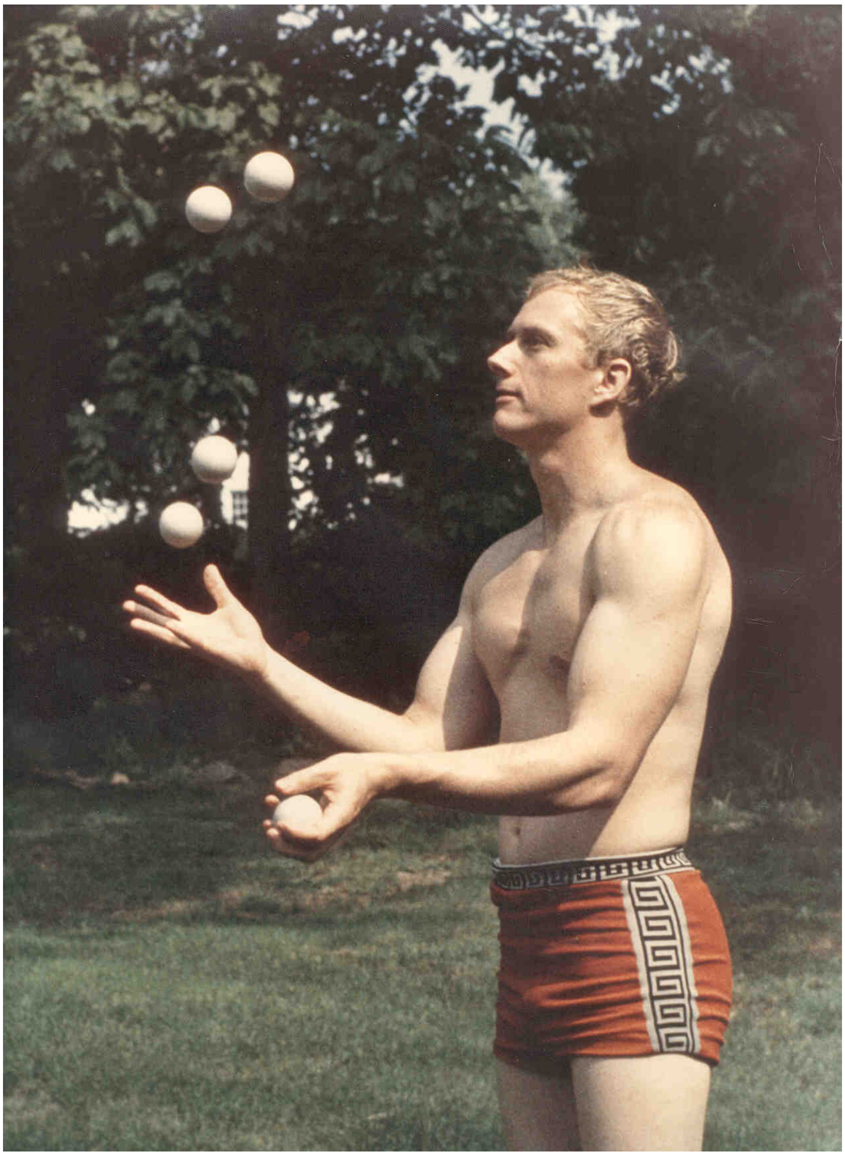

Figure 1: Ron 70s

too. Ron aimed for substantial and hard, yet concrete problems. He was a problem killer with an easy style. I still hear his "take it easy Jarik" - how helpful this was! He surrounded himself with very good people and aimed for depth and quality. In fact he was a very *concrete mathematician* in the style of the famous textbook [6].

I have been fortunate to work with Ron on papers and books mostly related to Ramsey's theorem and its variations. Ramsey's theorem is a universal mathematical principle often summarized by Ron as "complete disorder is impossible". This was perhaps Ron's favourite if not key area. In fact during his time *Ramsey theory* emerged as a "theory" from a mere particular collection of statements of "Ramsey-type" (due to Van den Waerden, Schur, Hilbert, Rado and others). In this development had the above Keszthely meeting an important crystallizing role and Ron Graham's influence was pivotal. This was particularly true for structural Ramsey theory where the starting group of researchers was of course small. See the preface and the selection of topics covered by [4], the book which became a standard reference for this emerging field.

Figure 2: Old Ramsey group: from left Rothschild, Deuber, Erdős, Voigt, Prömel, Graham, Nešetřil, Rödl (1981 Eger, blurred photo by R. Guy).

In this development a particular place was assumed by the Hales–Jewett theorem [10] and the Graham–Rothschild theorem [2]. These are strong statements which found many applications and serve as a tool for proving many Ramsey-type statements. Particularly they led to a solution of Rota's conjecture (which is the analog of Ramsey's theorem for finite vector spaces) by Graham, Leeb and Rothschild [3]. All five people involved in these early results received the inaugural Pólya Prize in 1971.

These results led to many papers since and blossomed into the whole theory. Today we seem to be witnessing a renaissance of the field in the context of topological dynamics, functional analysis, model theory and, of course, combinatorics. In 2016 there was even a meeting celebrating 50 years of Hales-Jewett theorem in Bellingham.

I cannot resist the temptation to try to outline the mathematical meaning of these results. Ramsey's theorem guarantees certain regularity in large structures. For graphs this regularity is a complete graph or an empty graph. Ramsey's theorem is in fact a general combinatorial principle useful across mathematics and the theory of computing. Some 50 years later Hales–Jewett and Graham–Rothschild found another such principle, this time both combinatorial and geometrical. It is possible to sketch it as follows:

Think of a finite set A as an alphabet, for example $A = \{1, 2, \ldots, k\}$.

Figure 3: 4 out of 5 winners of Pólya Prize in 1971 (photo curtesy J. Solymosi).

The product set A^d is then just a set of vectors $(a_1, \ldots, a_d)$ with each $a_i \in A$. Alternatively we may view A^d as a geometric object: A^d is $d-$dimensional cube (or rather $A-$cube) with sides indexed by A. Thus $\{1, 2, 3\}^3$ is the popular Rubik's cube, $\{1, 2, 3\}^2$ is a square lattice like in the game tic-tac-toe. In this way A^d may be viewed as a board for a $d-$dimensional version of this game. (In fact this was one of the motivations of the original paper [10]. As in tic-tac-toe we are looking for lines, horizontal, vertical, diagonal and this may be defined for any $d-$dimensional cube and more generally we can speak about $d-$dimensional sub-cubes of a $D-$dimensional cube. One can express lines and $d-$dimensional sub-cubes concisely as *parameter words* (a term coined by Graham–Rothschild) where parameters indicate which coordinates are "moving" . (In a square grid the lines have the form $(a\lambda), (\lambda b)$ and $(\lambda\lambda)$ for the diagonal.) The exact definition is a bit technical but it confirms the above intuition. And this is all that is needed in order to state the result of Graham and Rothschild [2].

Theorem For every choice of alphabet A and positive integers d, n there

Figure 4: H-J and G-R some 45 years later.

exists $N = N(A, d, n)$ such that whenever the set of all $d-$dimensional sub-cubes of A^N is partitioned in two parts then one of the parts has to contain an $n-$dimensional $A-$sub-cube with all its $d-$dimensional $A-$sub-cubes belonging to one of the classes of the partition.

(Recall, that Ramsey's theorem speaks about subsets instead of sub-cubes. Hales–Jewett theorem corresponds to $d = 0, n = 1$.)

It is perhaps surprising that such a seemingly technical result plays such an important role. But this is like Ramsey's theorem itself: it is a combinatorial principle which fits in diverse situations and assumptions. The Graham–Rothschild theorem is a far-reaching generalization of Ramsey's theorem, provides a proper setting for Van der Waerden's theorem and, as it was realized later, it yields a "dual" form of Ramsey's theorem. This inspiration lives on.

The mathematics of Ron Graham is important and it spans many diverse areas. But still I think that Ramsey theory was closest to his heart. It was also the topic of Ron's invited lecture at ICM 82 (held in Warsaw 1983), [5]. Ramsey theory was also dear to Paul Erdős as witnessed by the 2-volume set Mathematics of Paul Erdős where it occupies a whole chapter ([7, 8] , see also [9]). In fact these volumes, which were assembled still under the guidance of Paul Erdős himself, contain many pages written by editors reflecting a long experience of collaboration with Erdős.

The other parts of Ron Graham's activity are reflected by a forthcoming

Figure 5: 50 years of H-J (Bellingham 2016 - Photos courtesy J. Solymosi).

volume of the journal INTEGERS and also by volumes which were published for his 80th birthday [12, 1]. Ron was public figure and a well-known mathematician, often representing mathematics as a whole. This was nicely documented recently by an article in The New Yorker [13]. But I want to add yet another aspect of Ron's personality. I believe Ron Graham was a patriot. Patriot in a very good and decent sense. He liked very much Bell Labs; he liked his country. Perhaps this was one of the factors why he had such a keen interest in the development of friendship on the other side of the Iron Curtain. This interest was of course motivated by mathematics and it was forged by P. Erdős and the excellence of Hungarian combinatorics. But there was much more on a personal and, yes, human level - he really tried to be helpful. He encouraged us and served as a bridge to the world. And this was in those times when there were not many bridges at all, and it needed courage. It would take too long to illustrate this. Let us just mention that he helped to establish DIMATIA (as a "European DIMACS"), steadily invited people to Bell Labs and communicated about chances, possibilities and simply was spreading informations and books.

There were no obstacles or curtains for Ron. In this he is a great role model and this is the lasting legacy of his personality. He is and will be remembered by many.

Figure 6: Elegant Easy Style (photo by John Gimbel).

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